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CONNECTIONS IN POISSON GEOMETRY I: HOLONOMY AND INVARIANTS

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Abstract

We discuss contravariant connections on Poisson manifolds. For vector bundles, the corresponding operational notion of a contravariant derivative had been introduced by I. Vaisman. We show that these connections play an important role in the study of global properties of Poisson manifolds and we use them to define Poisson holonomy and new invariants of Poisson manifolds.

Introduction

Let M be a Poisson manifold and suppose that we require the existence of a linear connection on M, compatible with the Poisson tensor Π. Since parallel transport preserves the rank of the Poisson tensor, the Poisson manifold must be regular in order for such a connection to exist. Therefore, the usual notion of a covariant connection is not appropriate for the study of Poisson manifolds, as some of the most interesting examples of Poisson manifolds are non-regular. For non-regular Poisson manifolds the symplectic foliation is singular and the dimension of the leaves varies, so one can only hope to compare tangent spaces at different points of the same symplectic leaf.

One possible way around this difficulty is to use families of connections parameterized by the leaves. However, there are examples showing that the symplectic foliation can be wild, so the space of leaves will not be easy to parameterize.

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A much more efficient and direct approach, to be introduced in this paper, is through the notion of a contravariant connection, a concept that mimics for the case of Poisson manifolds the usual notion of a covariant connection.

Assume we are given a principal bundle over a manifold M:

$$\begin{tikzcd} P \arrow[loop right, "G"] \arrow[d, "p"'] \\ M \end{tikzcd}$$

then a covariant connection Γ on this principal bundle is defined by a G-invariant horizontal distribution $u \mapsto H_u$ in P. Given a connection Γ, we have a notion of *horizontal lift*: $h(u,v) \in T_uP$ is the unique tangent vector to H_u which projects to the vector $v \in T_{p(u)}M$. Conversely, the horizontal lift h defines the horizontal distribution $H_u = \left\{h(u,v) : v \in T_{p(u)}M\right\}$, so h completely determines the connection.

We shall define a contravariant connection on a principal bundle over a Poisson manifold by defining analogously the horizontal lift of cotangent vectors. To formulate this notion, observe that h is defined precisely for pairs (u,v) in p^*TM, the pullback bundle by p of the tangent bundle over M. Denote by $\widehat{p} : p^*TM \to TM$ the induced bundle map so we have the commutative diagram

$$\begin{tikzcd} p^*TM \arrow[r, "\widehat{p}"] \arrow[d, "\widehat{\pi}"'] & TM \arrow[d, "\pi"] \\ P \arrow[r, "p"'] & M \end{tikzcd}$$

Then we can define a covariant connection to be a bundle map $h : p^*TM \to TP$, such that:

(CI) h is horizontal, i. e., the following diagram commutes:

$$\begin{tikzcd} p^*TM \arrow[r, "h"] \arrow[d, "\widehat{p}"'] & TP \arrow[d, "p_*"] \\ TM \arrow[r, "\mathrm{id}"'] & TM \end{tikzcd}$$

(CII) h is G-invariant: $h(ua,v) = (R_a)_* h(u,v)$, for all $a \in G$;

Assume now that M is a Poisson manifold. According to a general philosophical principle, in Poisson geometry sometimes the cotangent bundle plays the role of the tangent bundle. Hence, we replace TM by T^*M in the diagrams above, whenever it makes sense. Thus we are lead to the notion of a *contravariant connection* on a Poisson manifold: this is a bundle map $h : p^*T^*M \to TP$, such that:

(CI)* The following diagram commutes:

$$\begin{tikzcd} p^*T^*M \arrow[r, "h"] \arrow[d, "\widehat{p}"'] & TP \arrow[d, "p_*"] \\ T^*M \arrow[r, "\#"'] & TM \end{tikzcd}$$

where $\# : T^*M \to TM$ is the bundle map induced by the Poisson tensor;

(CII)* h is G-invariant: $h(ua, \alpha) = (R_a)_* h(u, \alpha)$, for all $a \in G$;

Given a point x in M and a covector $\alpha \in T^*_x M$, the vector $h(u, \alpha) \in T_u P$ will be called the *horizontal lift* of α to the point u in the fiber over x. On any fibration one can also consider *generalized contravariant connections* which satisfy only (CI)*.

With such a definition at hand one can then develop the usual concepts of parallelism, curvature, holonomy, geodesic, etc. In particular, for a contravariant connection on a vector bundle $p : E \to M$, one obtains in a way entirely analogous to the covariant case, the notion of a *contravariant derivative* operator D: for each 1-form α on M, D_α maps sections of E to sections of E and satisfies

i) $D_{\alpha+\beta}\phi = D_\alpha\phi + D_\beta\phi$;

ii) $D_\alpha(\phi + \psi) = D_\alpha\phi + D_\alpha\psi$;

iii) $D_{f\alpha} = f D_\alpha\phi$;

iv) $D_\alpha(f\phi) = f D_\alpha\phi + \#\alpha(f)\phi$;

where $\alpha, \beta \in \Omega^1(M)$, ϕ, ψ are sections of E, and $f \in C^\infty(M)$. Conversely, every such operator is induced by a contravariant connection. Moreover, one can show that there always exists a linear connection preserving the Poisson tensor. In [11] Vaisman introduces the notion of contravariant derivative using i)-iv) as axioms.

In spite of its formal similarities with covariant connections, there are striking differences in contravariant Poisson geometry. For example, the holonomy of a connection may be non-discrete when the connection is flat, contravariant connections cannot be pushed back or forward, etc. However, just like in ordinary geometry, contravariant connections are useful to study global properties of Poisson manifolds.

Recall that the local structure of a Poisson manifold is given by the Weinstein splitting theorem, also known as the generalized Darboux theorem (see [13], Thm. 2.1). In a neighborhood of a point, the Poisson structure splits as a direct product of a symplectic structure and a Poisson structure which vanishes at the point. So on the normal space to each symplectic leaf we have a notion of *transverse Poisson structure*.

In global Poisson geometry one would like to understand the geometry and topology of the symplectic foliation. Using generalized contravariant connections we show that we have a notion of *Poisson holonomy* of the symplectic foliation, analogous to the holonomy in the theory of regular foliations. The corresponding linear holonomy coincides with the *linear Poisson holonomy* introduced by Ginzburg and Golubev in [4]. The Poisson holonomy map is by Poisson automorphisms of the transverse Poisson structure.

Poisson holonomy is not homotopy invariant, but factoring out the inner Poisson automorphisms one obtains a notion of *reduced Poisson holonomy* invariant by homotopy, and we can prove the following analogue of the Reeb stability theorem:

Theorem. *Let S be a compact, transversely stable leaf, with finite reduced Poisson holonomy. Then S is stable, i. e., S has arbitrarily small neighborhoods which are invariant under all hamiltonian automorphisms. Moreover, each symplectic leaf of M near S is a bundle over S whose fiber is a finite union of symplectic leaves of the transverse Poisson structure.*

We also discuss another related notion of holonomy, which we call *strict Poisson holonomy*, and which allows one to discuss global splitting of an entire neighborhood of a symplectic leaf. The corresponding stability theorem states that if S is transversely stable and has finite strict Poisson holonomy, there is a neighborhood of S which is Poisson covered by a product $\tilde{S} \times N$, with $\tilde{S}$ a finite cover of S.

Linear Poisson holonomy in turn can be discussed from the point of view of linear contravariant connections and, for each symplectic leaf, there is a notion of Bott contravariant connection. For a non-regular

Poisson manifold, we do not have a normal bundle (over the whole of M) to the symplectic foliation. However, there is an appropriate notion of a basic connection on M: these are linear contravariant connections which preserve the Poisson tensor and restrict in each leaf to the Bott contravariant connection. Comparing a basic connection to a riemannian connection one is lead to "exotic" or secondary Poisson characteristic classes. These are Poisson cohomology classes which give information on both the Poisson geometry and the topology of the symplectic foliation of M. In degree 1, this class actually coincides with the *modular class* of M. This invariant was discussed recently by Weinstein in [12], where he shows that the modular class is an obstruction to the existence of measures in M invariant under the hamiltonian flows.

As a final note we remark that the most general setup for contravariant connections is in the context of Lie algebroids. Although we have omitted any references to Lie Algebroids, the results discussed here should go through without any major changes, and this will be discussed elsewhere.

In a follow up to this paper ([3]) we will discuss invariant connections.

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1. Contravariant connections on principal bundles

1.1 Contravariant cartan calculus

On a Poisson manifold there is a calculus on contravariant objects, analogous to the usual Cartan calculus on differential forms. We recall here some of the formulas and fix notation and conventions for later use. Proofs of the results stated in this introductory paragraph can be found in Vaisman's monograph [10].

Let M be a Poisson manifold and denote by $\Pi \in \mathcal{X}^2(M)$([1]) the Poisson bivector field, so the Poisson bracket on M is given by

$$\{f_1, f_2\} = \Pi(df_1, df_2), \qquad f_1, f_2 \in C^\infty(M). \tag{1.1}$$

We also have a bundle map $\# : T^*M \longrightarrow TM$ defined by

$$\beta(\#\alpha) = \Pi(\alpha, \beta), \qquad \alpha, \beta \in T^*M. \tag{1.2}$$

On the space of differential 1-forms $\Omega^1(M)$ the Poisson tensor induces a Lie bracket

$$[\alpha, \beta] = \mathcal{L}_{\#\alpha}\beta - \mathcal{L}_{\#\beta}\alpha - d(\Pi(\alpha, \beta)), \qquad \alpha, \beta \in \Omega^1(M), \tag{1.3}$$

and for this Lie bracket and the usual Lie bracket on vector fields, the map $\# : \Omega^1(M) \longrightarrow \mathcal{X}^1(M)$ is a Lie algebra homomorphism:

$$\#[\alpha, \beta] = [\#\alpha, \#\beta]. \tag{1.4}$$

We denote as usual by $X_f = \#(df)$ the hamiltonian vector field associated with the function $f \in C^\infty(M)$, and we have

$$[\alpha, f\beta] = f[\alpha, \beta] + \#\alpha(f)\beta = f[\alpha, \beta] - \left(i_{X_f}\alpha\right)\beta. \tag{1.5}$$

The existence of a Lie bracket on the space of 1-forms allows one to mimic the algebraic definitions of d, i_X and $\mathcal{L}_X$, to obtain contravariant versions of these operators.

First, one defines the contravariant exterior differential

$$\delta : \mathcal{X}^r(M) \longrightarrow \mathcal{X}^{r+1}(M)$$

by:

$$\begin{aligned} &\delta Q(\alpha_0, \dots, \alpha_r) \\ &\quad = \frac{1}{r+1}\sum_{k=0}^{r}(-1)^{k+1}\#\alpha_k(Q(\alpha_0, \dots, \widehat{\alpha}_k, \dots, \alpha_r)) \\ &\qquad + \frac{1}{r+1}\sum_{k<l}(-1)^{k+l+1}Q([\alpha_k, \alpha_l], \alpha_0, \dots, \widehat{\alpha}_k, \dots, \widehat{\alpha}_l, \dots, \alpha_r). \end{aligned} \tag{1.6}$$

[1] We denote by $\Omega^r(M)$ and $\mathcal{X}^r(M)$, respectively, the spaces of differential r-forms and r-multivector fields on a manifold M.

where $\alpha_0, \dots, \alpha_r \in \Omega^1(M)$. This differential satisfies:

$$\delta^2(Q) = 0, \tag{1.7}$$

$$\delta(Q_1 \wedge Q_2) = \delta Q_1 \wedge Q_2 + (-1)^{\deg Q_1} Q_1 \wedge \delta Q_2. \tag{1.8}$$

Moreover, if we extend the definition of # to forms of any degree by setting

$$\#\lambda(\alpha_1, \dots, \alpha_r) = (-1)^r \lambda(\#\alpha_1, \dots, \#\alpha_r), \tag{1.9}$$

we have

$$\delta(\#\lambda) = \#(d\lambda). \tag{1.10}$$

The cohomology associated with δ is called the *Poisson cohomology* of M and is denoted by $H^*_\Pi(M)$. This relation shows that there is a homomorphism from de Rham cohomology to Poisson cohomology $\# : H^*(M) \to H^*_\Pi(M)$, which in the case of a symplectic manifold is an isomorphism.

Next, for each form $\alpha \in \Omega^1(M)$ there is an operator of contraction by α, denoted $i_\alpha : \mathcal{X}^r(M) \longrightarrow \mathcal{X}^{r-1}(M)$, and an operator of Lie derivative in the direction of α, denoted $\mathcal{L}_\alpha : \mathcal{X}^r(M) \longrightarrow \mathcal{X}^r(M)$, given by

$$(i_\alpha Q)(\alpha_1, \dots, \alpha_{r-1}) = Q(\alpha, \alpha_1, \dots, \alpha_{r-1}), \tag{1.11}$$

$$\begin{aligned}(\mathcal{L}_\alpha Q)(\alpha_1, \dots, \alpha_r) &= \#\alpha(Q(\alpha_1, \dots, \alpha_r)) \\ &\quad - \sum_{k=1}^{r} Q(\alpha_1, \dots, [\alpha, \alpha_k], \dots, \alpha_r).\end{aligned} \tag{1.12}$$

We have formulas analogous to the usual formulas from Cartan calculus:

$$i_{[\alpha,\beta]} = \mathcal{L}_\alpha i_\beta - i_\beta \mathcal{L}_\alpha, \tag{1.13}$$

$$\mathcal{L}_{[\alpha,\beta]} = \mathcal{L}_\alpha \mathcal{L}_\beta - \mathcal{L}_\beta \mathcal{L}_\alpha, \tag{1.14}$$

$$\mathcal{L}_\alpha = i_\alpha \delta + \delta i_\alpha, \tag{1.15}$$

$$\delta \mathcal{L}_\alpha = \mathcal{L}_\alpha \delta. \tag{1.16}$$

In fact, the musical homomorphism relates these operators to the usual ones, so for every 1-form $\alpha \in \Omega^1(M)$, every r-form $\lambda \in \Omega^r(M)$ and every r-multivector field $Q \in \mathcal{X}^r(M)$, one has:

$$i_\alpha(\#\lambda) = (-1)^r \#(i_{\#\alpha}\lambda), \tag{1.17}$$

$$\mathcal{L}_\alpha(\#\lambda) = (-1)^r \#(\mathcal{L}_{\#\alpha}\lambda), \tag{1.18}$$

$$\mathcal{L}_{df} Q = \mathcal{L}_{X_f} Q. \tag{1.19}$$

We can also extend $\mathcal{L}_\alpha$ to the exterior algebra $\Omega^*(M)$ by setting

$$\mathcal{L}_\alpha\beta = [\alpha, \beta], \qquad \beta \in \Omega^1(M), \tag{1.20}$$

and requiring $\mathcal{L}_\alpha$ to preserve type and act as a derivation. Finally, we recall that the contravariant differential can also be defined by

$$\delta Q = -[\Pi, Q]_s, \tag{1.21}$$

where $[\ ,\ ]_s$ denotes the Schouten bracket.

1.2 Contravariant connections

Let $P(M,G)$ be a smooth principal bundle over a Poisson manifold M with structure group G. We let $p : P \to M$ be the projection, and for each $u \in P$ we denote by $G_u \subset T_u(P)$ the subspace consisting of vectors tangent to the fiber through u. If we denote by p^*T^*M the pullback bundle, so there is a bundle map $\widehat{p} : p^*T^*M \to T^*M$ which makes the following diagram commutative

$$\begin{tikzcd} p^*T^*M \arrow[r, "\widehat{p}"] \arrow[d, "\widehat{\pi}"'] & T^*M \arrow[d, "\pi"] \\ P \arrow[r, "p"'] & M \end{tikzcd}$$

where on the vertical arrows we have the canonical projections. Recalling that $p^*T^*M = \{(u,\alpha) \in P \times T^*M : p(u) = \pi(\alpha)\}$, we see that we have a natural right G-action on p^*T^*M defined by $(u,\alpha)\cdot a \equiv (ua,\alpha)$, if $a \in G$.

Definition 1.2.1. A CONTRAVARIANT CONNECTION Γ in $P(M,G)$ is a smooth bundle map $h : p^*T^*M \to TP$, such that:

(CI)* The following diagram commutes:

$$\begin{tikzcd} p^*T^*M \arrow[r, "h"] \arrow[d, "\widehat{p}"'] & TP \arrow[d, "p_*"] \\ T^*M \arrow[r, "\#"'] & TM \end{tikzcd}$$

(CII)* h is G-invariant: $h(ua,\alpha) = (R_a)_*h(u,\alpha)$, for all $a \in G$;

Given $(u, \alpha) \in p^*T^*M$, we call the vector $h(u, \alpha) \in T_uP$ the *horizontal lift* of the 1-form α to u. The subspace of T_uP formed by all such horizontal vectors is denoted by $\mathcal{H}_u$. The assignment $u \mapsto \mathcal{H}_u$ is a smooth, generalized, distribution on P called the *horizontal distribution* of the connection (by "smooth" we mean that for each point $u_0 \in P$ there exists a neighborhood $u_0 \in U \subset P$ and smooth vector fields $X_1, \ldots, X_r$ in U, such that $\mathcal{H}_u = \operatorname{span}\{X_1|_u, \ldots, X_r|_u\}$ for all $u \in U$). Note that, as opposed to the covariant case, the rank of the horizontal distribution will vary, and that this distribution does not define the connection uniquely.

It follows from (CI)* in the definition of a contravariant connection, that the horizontal spaces $\mathcal{H}_u$ project onto the tangent space T_xS to the symplectic leaf S through $x = p(u)$. In general, we have neither $T_uP = G_u + \mathcal{H}_u$ nor $G_u \cap \mathcal{H}_u = \{0\}$. As usual, a vector $X \in T_uP$ will be called *vertical* (resp. *horizontal*), if it lies in G_u (resp. $\mathcal{H}_u$). If M is not symplectic, a vector does not split into a sum of an horizontal and a vertical component, so the usual definitions of lift of curves, connection form, etc., do not make sense in this context.

Later on, we shall need to consider *generalized contravariant connections*, by which mean that axiom (CII)* need not be satisfied. Of course, such connections can be considered on any fibration over a Poisson manifold.

1.3 Connection vector fields

If $\mathfrak{g}$ is the Lie algebra of G, we can express a contravariant connection in P by a family of $\mathfrak{g}$-valued vector fields, each defined in an open subset of M. One should have in mind that, in this theory, multivector fields play the role of differential forms.

Henceforth, we use the following notation: We denote by $\{U_j\}$ an open cover of M, by $\psi_j : p^{-1}(U_j) \to U_j \times G$ a family of trivializing isomorphisms, and by $\psi_{jk} : U_j \cap U_k \to G$ the associated transition functions. For each j, we let $s_j : U_j \to P$ be the section over U_j defined by $s_j(x) = \psi_j^{-1}(x, e)$, where $e \in G$ is the identity.

On each open set U_j we define a $\mathfrak{g}$-valued vector field Λ_j as follows: if $\alpha \in \Omega^1(U_j)$, $x \in U_j$, and $u = s_j(x)$, then

$$X_u = (s_j)_* \# \alpha_x - h(s_j(x), \alpha_x) \in T_uP$$

is a vertical vector since, by (CI)*, we have:

$$p_* X_u = p_* \cdot (s_j)_* \# \alpha_x - p_* h(s_j(x), \alpha_x) = \# \alpha_x - \# \alpha_x = 0.$$

We let $\Lambda_j(\alpha)_x$ be the unique element $A \in \mathfrak{g}$ such that $X_u = \sigma(A)_u$, which exists by (CII)*. The $\{\Lambda_j\}$ are called the *connection vector fields* of the contravariant connection Γ.

In order to state the transformation rule for the connection vector fields, it is convenient to introduce the following notation: if $\phi : M \to N$ is a smooth map defined on a Poisson manifold M its *contravariant differential* is the bundle map $\delta\phi : T^*M \to TN$ defined by:

$$\delta\phi(\alpha_x) = d_x\phi \cdot \#\alpha_x, \qquad \alpha_x \in T^*_x M. \tag{1.22}$$

If $N = \mathbb{R}$ this notation is consistent with the contravariant differential introduced above, if we think of 0-vector fields as functions.

Proposition 1.3.1. *The connection vector fields $\{\Lambda_j\}$ are related by*

$$\Lambda_k = Ad(\psi_{jk}^{-1})\Lambda_j + \psi_{jk}^{-1}\delta\psi_{jk}, \quad \text{on } U_j \cap U_k. \tag{1.23}$$

Conversely, given a family of $\mathfrak{g}$-valued vector fields, each defined in U_j, satisfying relations (1.23), there is a unique contravariant connection in $P(M,G)$ which gives rise to the $\{\Lambda_j\}$.

Proof. Given a contravariant connection, define the vector fields $\{\Lambda_j\}$ as above. If $U_j \cap U_k$ is non-empty, we have $s_k(x) = s_j(x)\psi_{jk}(x)$, for all $x \in U_j \cap U_k$. If we set $a = \psi_{jk}(x) \in G$, it follows from Leibniz rule that

$$s_{k*}(X) = (R_a)_*(s_j)_*(X) + \sigma((L_{a^{-1}})_* \cdot (\psi_{jk})_*X). \tag{1.24}$$

If we compute both sides on $X = \#\alpha$, we obtain

$$\begin{aligned}
\sigma(\Lambda_k(\alpha))_{ua} &= s_{k*}(\#\alpha)_{ua} - h(ua,\alpha)\\
&= (R_a)_*(s_j)_*(\#\alpha)_u + \sigma((L_{a^{-1}})_* \cdot (\psi_{jk})_*\#\alpha)_u - (R_a)_*h(u,\alpha)\\
&= (R_a)_*\sigma(\Lambda_j(\alpha))_u + \sigma((L_{a^{-1}})_* \cdot (\psi_{jk})_*\#\alpha)_u\\
&= \sigma(\mathrm{Ad}(\psi_{jk}^{-1})\Lambda_j(\alpha))_u + \sigma(\psi_{jk}^{-1}\delta\psi_{jk}(\alpha))_u.
\end{aligned}$$

as required.

Conversely, given a family of $\mathfrak{g}$-valued vector fields satisfying relations (1.23), we define a contravariant connection Γ by letting the horizontal lift be defined by

$$h(u,\alpha) = s_{j_*}(\#\alpha)_u - \sigma(\Lambda_j(\alpha))_u, \tag{1.25}$$

whenever s_j is a section with $s_j(x) = u$. If s_k is another section with $s_k(x) = u$, it follows from (1.23) and (1.24), with $\psi_{jk}(x) = a(x) = e$, that

$$\begin{aligned} s_{k*}(\#\alpha)_u - \sigma(\Lambda_k(\alpha))_u &= s_{j*}(\#\alpha)_u + \sigma((\psi_{jk})_*\#\alpha)_u - \sigma(\Lambda_k(\alpha))_u \\ &= s_{j*}(\#\alpha)_u - \sigma(\Lambda_j(\alpha))_u, \end{aligned}$$

so this definition is independent of the section used. Conditions (CI)* of the definition is easily verified. As for (CII)*, we note that if $\psi_{jk}(x) = a \in G$ is constant, then $\Lambda_k = \mathrm{Ad}(a^{-1})\Lambda_j$ and equation (1.24) gives $s_{k*}(X) = (R_a)_*(s_j)_*(X)$. Therefore, for any 1-form α, we find

$$\begin{aligned} h(ua, \alpha) &= s_{k*}(\#\alpha)_{ua} - \sigma(\Lambda_k(\alpha))_{ua} \\ &= (R_a)_* s_{j*}(\#\alpha)_u - \sigma(Ad(a^{-1})\Lambda_j(\alpha))_{ua} \\ &= (R_a)_* h(u, \alpha), \end{aligned}$$

as wished. q.e.d.

1.4 Curvature

For a contravariant connection Γ with family of connection vector fields $\{\Lambda_j\}$ we define a corresponding family of *curvature bivector fields* $\{\Xi_j\}$ by:

$$\Xi_j = \delta\Lambda_j + \frac{1}{2}[\Lambda_j, \Lambda_j]. \tag{1.26}$$

Here, we are using the notation $[\xi, \zeta]$ for the $\mathfrak{g}$-valued multivector field defined by

$$[\xi, \zeta] = \sum_{a,b,c} C^a_{bc}\xi^b \wedge \zeta^c \mathbf{e}_a,$$

where $\xi = \sum_a \xi^a \mathbf{e}_a$ and $\zeta = \sum_a \xi^a \mathbf{e}_a$ are $\mathfrak{g}$-valued multivector fields, relative to a basis $\{\mathbf{e}_a\}$ for $\mathfrak{g}$, and C^a_{bc} are the structure constants of $\mathfrak{g}$ relative to the same basis.

Proposition 1.4.1. *The curvature bivector fields of a contravariant connection are related by*

$$\Xi_k = Ad(\psi_{jk}^{-1})\Xi_j, \quad \textit{on } U_j \cap U_k. \tag{1.27}$$

Moreover, they satisfy the Bianchi identity:

$$\delta\Xi_j + [\Lambda_j, \Xi_j] = 0. \tag{1.28}$$

Proof. Set $\eta_{jk} = \psi_{jk}^{-1}\delta\psi_{jk}$. Then we have the "Maurer-Cartan equations"

$$\delta\eta_{jk} = -\frac{1}{2}[\eta_{jk}, \eta_{jk}]. \tag{1.29}$$

On the other hand, if Λ_k and Λ_j are related by (1.23) we find, using (1.8),

$$\begin{aligned}\delta\Lambda_k &= \delta(Ad(\psi_{jk}^{-1})\Lambda_j) + \delta\eta_{jk}\\ &= Ad(\psi_{jk}^{-1})\delta\Lambda_j - \frac{1}{2}[\eta_{jk}, Ad(\psi_{jk}^{-1})\Lambda_j] + \frac{1}{2}[Ad(\psi_{jk}^{-1})\Lambda_j, \eta_{jk}] + \delta\eta_{jk}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\delta\Lambda_k + \frac{1}{2}[\Lambda_k, \Lambda_k] &= Ad(\psi_{jk}^{-1})\delta\Lambda_j + \frac{1}{2}[Ad(\psi_{jk}^{-1})\Lambda_j, Ad(\psi_{jk}^{-1})\Lambda_j]\\ &= Ad(\psi_{jk}^{-1})\left(\delta\Lambda_j + \frac{1}{2}[\Lambda_j, \Lambda_j]\right).\end{aligned}$$

so (1.27) holds.

Bianchi's identity (1.28) follows from $\delta^2\Lambda_j = 0$ and the derivation property (1.8) of δ. q.e.d.

Remark 1.4.2. The structure equation (1.26) and the Bianchi identity (1.28) show that one should think of the operator $\delta + [\Lambda_j, \cdot]$ as a *contravariant derivative* acting on $\mathfrak{g}$-valued multivector fields. This comment will be made precise later.

It follows from (1.27), that given 1-forms $\alpha, \beta \in \Omega^1(M)$, we can define a $\mathfrak{g}$-valued function $\Xi(\alpha, \beta)$ in P by:

$$\Xi(\alpha, \beta)_{s_j(x)} \equiv \Xi_j(\alpha, \beta).$$

$\Xi(\alpha, \beta)$ gives the following geometric interpretation of the curvature: Given a 1-form $\alpha \in \Omega^1(M)$, denote by $h(\alpha)$ the horizontal lift of α, so $h(\alpha)_u = h(u, \alpha)$ and

$$u \mapsto \mathcal{H}_u = \left\{h(\alpha)_u : \alpha \in \Omega^1(M)\right\}$$

is the horizontal distribution.

Proposition 1.4.3. *Let $\alpha, \beta \in \Omega^1(M)$. Then:*

$$[h(\alpha), h(\beta)] - h([\alpha, \beta]) = -2\sigma(\Xi(\alpha, \beta)), \tag{1.30}$$

To prove the proposition we need the following lemma:

Lemma 1.4.4. *For any* $\alpha, \beta \in \Omega^1(U_j)$

$$[h(\alpha), \sigma(\Lambda_j(\beta))] = -\sigma(\#\alpha(\Lambda_j(\beta))). \tag{1.31}$$

Proof. The flux of the vector field $\sigma(\Lambda_j(\beta))$ is

$$\Phi_t(u) = u \exp(t\Lambda_j(\beta)(p(u))),$$

so we have:

$$[h(\alpha), \sigma(\Lambda_j(\beta))]_{u_0} = -\lim_{t\to 0} \frac{1}{t} \left(h(u_0, \alpha) - d\Phi_{-t} \cdot h(\Phi_t(u_0), \alpha)\right).$$

But:

$$\begin{aligned} d\Phi_{-t} \cdot h(\Phi_t(u_0), \alpha) &= dR_{\exp(t\Lambda_j(\beta)(p(u)))} \cdot h(\Phi_t(u_0), \alpha) \\ &\quad + d\Psi \cdot h(\Phi_t(u_0), \alpha) \\ &= h(u_0, \alpha) + d\Psi \cdot h(\Phi_t(u_0), \alpha), \end{aligned}$$

where $\Psi(u) = u_0 \exp(t\Lambda_j(\beta)(p(u)))$. Let $s \mapsto \tilde{\gamma}(s,t)$ be the integral curve of $h(\alpha)$ through $\Phi_t(u_0)$. Then $s \mapsto \gamma(s,t) = p(\tilde{\gamma}(s,t))$ is an integral curve of $\#\alpha$, and we have:

$$d\Psi \cdot h(\Phi_t(u_0), \alpha) = \frac{d}{ds}\, u_0 \exp(t\Lambda_j(\beta)(\gamma(s,t)))\big|_{s=0}.$$

We conclude that

$$\begin{aligned} [h(\alpha), \sigma(\Lambda_j(\beta))]_{u_0} &= -\frac{d}{dt}\left[\frac{d}{ds}\, u_0 \exp(t\Lambda_j(\beta)(\gamma(s,t)))\big|_{s=0}\right]_{t=0} \\ &= \frac{d}{ds}\, \sigma(\Lambda_j(\beta)(\gamma(s,0)))_{u_0}\big|_{s=0} \\ &= \sigma(\#\alpha(\Lambda_j(\beta))_{p(u_0)})_{u_0}, \end{aligned}$$

and the lemma follows. q.e.d.

Proof of Proposition 1.4.3. Over U_j we have

$$(s_j)_*\#\alpha = \sigma(\Lambda_j(\alpha)) + h(\alpha),$$

so we find:

$$\begin{aligned}[h(\alpha), h(\beta)] &= (s_j)_*\#[\alpha,\beta] - [(s_j)_*\#\alpha, \sigma(\Lambda_j(\beta))] \\ &\quad - [\sigma(\Lambda_j(\alpha)), (s_j)_*\#\beta] + [\sigma(\Lambda_j(\alpha)), \sigma(\Lambda_j(\beta))] \\ &= h([\alpha,\beta]) + \sigma(\Lambda_j([\alpha,\beta])) - [h(\alpha), \sigma(\Lambda_j(\beta))] \\ &\quad - [\sigma(\Lambda_j(\alpha)), h(\beta)] - [\sigma(\Lambda_j(\alpha)), \sigma(\Lambda_j(\beta))]) \\ &= h([\alpha,\beta]) + \sigma(\Lambda_j([\alpha,\beta]) - \#\alpha(\Lambda_j(\beta)) + \#\beta(\Lambda_j(\alpha)) \\ &\quad - \sigma([\Lambda_j(\alpha), \Lambda_j(\beta)]) \\ &= h([\alpha,\beta]) - \sigma(2\delta\Lambda_j(\alpha,\beta) + [\Lambda_j,\Lambda_j](\alpha,\beta)) \\ &= h([\alpha,\beta]) - \sigma(2\Xi_j(\alpha,\beta)).\end{aligned}$$

q.e.d.

By a *flat contravariant connection* we shall mean a connection whose horizontal distribution is integrable.

Proposition 1.4.5. *A contravariant connection is flat iff its curvature bivector fields vanish.*

Proof. By a result of Hermann [6], a generalized distribution associated with a vector subspace $\mathcal{D} \subset \mathcal{X}(M)$ is integrable iff it is involutive and rank invariant. Taking $\mathcal{D} = \{h(df) : f \in C^\infty(M)\}$ so that $\mathcal{H}_u = \{X(u) : X \in \mathcal{D}\}$, Proposition 1.4.3 shows that $\mathcal{D}$ is involutive iff the curvature bivector fields vanish. Hence, all it remains to show is that if the curvature vanishes and $\gamma(t)$ is an integral curve of $h(df)$ then $\dim \mathcal{H}_{\gamma(t)}$ is constant, for all small enough t.

Let $\tilde{\Phi}_t$ be the flow of $h(df)$ and let $\Phi_t = p \circ \tilde{\Phi}_t$ be the flow of $\#df = X_f$. If $\alpha \in \Omega^1(M)$ we claim that

$$(\tilde{\Phi}_t)_* h(\alpha) = h(\Phi^*_{-t}\alpha),$$

for small enough t. In fact, the infinitesimal version of this relation is

$$[h(df), h(\alpha)] = h(\mathcal{L}_{X_f}\alpha) = h([df,\alpha]),$$

which by (1.30) holds, since we are assuming that the curvature vanishes.

Therefore, the flow $\tilde{\Phi}_t$ gives an isomorphism between $\mathcal{H}_{\gamma(0)}$ and $\mathcal{H}_{\gamma(t)}$, for small enough t, so $\mathcal{D}$ is rank invariant. q.e.d.

1.5 Parallelism and holonomy

Parallel displacement of fibers can be defined along curves *lying* on a symplectic leaf of M.

If $\gamma : [0,1] \to M$ is a smooth curve lying on a symplectic leaf S, then γ is also smooth as map $\gamma : [0,1] \to S$. This follows from the existence of "canonical coordinates" for M as given by the generalized Darboux theorem. Also, by the same theorem, we can choose a smooth family $t \mapsto \alpha(t) \in T^*M$ of covectors such that $\#\alpha(t) = \dot\gamma(t)$. Following [4], we shall call the pair $(\gamma(t), \alpha(t))$ a *cotangent curve.*

Proposition 1.5.1. *Let $(\gamma(t), \alpha(t))$ be a cotangent curve. For any u_0 in P with $p(u_0) = \gamma(0)$ there exists a unique horizontal lift $\tilde\gamma : [0,1] \to P$, which satisfies the system*

$$\begin{cases} \dot{\tilde\gamma}(t) = h(\tilde\gamma(t), \alpha(t)), \\ \\ \tilde\gamma(0) = u_0. \end{cases} \tag{1.32}$$

Proof. By standard results from the theory of o.d.e.'s with time dependent coefficients, system (1.32) has a unique maximal solution. We claim that this solution exists for all $t \in [0,1]$.

By local triviality of the bundle we can find a curve $\bar\gamma : [0,1] \to P$ with $\bar\gamma(0) = u_0$ and $p(\bar\gamma(t)) = \gamma(t)$. We look for a curve $a(t) \in G$, such that $\tilde\gamma(t) = \bar\gamma(t)a(t)$ satisfies (1.32). Differentiating, we have

$$\dot{\tilde\gamma}(t) = \dot{\bar\gamma}(t)a(t) + \bar\gamma(t)\dot a(t).$$

We therefore require $a(t)$ to satisfy the equation

$$\dot{\bar\gamma}(t)a(t) + \bar\gamma(t)\dot a(t) = h(\bar\gamma(t)a(t), \alpha(t)),$$

or, equivalently,

$$\bar\gamma(t)\dot a(t)a^{-1}(t) = h(\bar\gamma(t), \alpha(t)) - \dot{\bar\gamma}(t).$$

The right hand side of this equation belongs to $G_{\bar\gamma(t)}$ since

$$p_*(h(\bar\gamma(t), \alpha(t)) - \dot{\bar\gamma}(t)) = \#\alpha(t) - \frac{d}{dt}p(\bar\gamma(t)) = \#\alpha(t) - \dot\gamma(t) = 0.$$

Therefore, there exists some curve $A(t) : [0,1] \to \mathfrak{g}$ such that

$$\bar\gamma(t)\dot a(t)a^{-1}(t) = \bar\gamma(t)A(t).$$

Since the initial value problem

$$\dot{a}(t)a^{-1}(t) = A(t), \qquad a(0) = e,$$

always has a solution, defined wherever $A(t)$ is defined, our claim follows. q.e.d.

Now using the proposition we can define parallel displacement of the fibers along a cotangent curve $(\gamma(t), \alpha(t))$ in the usual form: if $u_0 \in p^{-1}(\gamma(0))$ we define $\tau(u_0) = \tilde{\gamma}(1)$, where $\tilde{\gamma}(t)$ is the unique horizontal lift of $(\gamma(t), \alpha(t))$ starting at u_0, We obtain a map $\tau : p^{-1}(\gamma(0)) \to p^{-1}(\gamma(1))$, which will be called *parallel displacement* of the fibers along the cotangent curve $(\gamma(t), \alpha(t))$. It is clear, since horizontal curves are mapped by R_a to horizontal curves, that parallel displacement commutes with the action of G:

$$\tau \circ R_a = R_a \circ \tau. \tag{1.33}$$

Therefore, parallel displacement is an isomorphism between the fibers.

If $x \in M$ lies in the symplectic leaf S, let $\Omega(S, x)$ be the loop space of S at x. Then for each cotangent loop (γ, α), with $\gamma \in \Omega(S, x)$, parallel displacement along (γ, α) gives a an isomorphism of the fiber $p^{-1}(x)$ into itself. The set of all such isomorphisms forms the holonomy group of Γ, with reference point x, and is denoted $\Phi(x)$. Similarly, one has the restricted holonomy group, with reference point x, denoted $\Phi^0(x)$, defined by using cotangent loops in S which are homotopic to the trivial loop.

If $u \in p^{-1}(x)$ then we can also define the holonomy groups $\Phi(u)$ and $\Phi^0(u)$. Just as in the covariant case, $\Phi(u)$ is the subgroup of G consisting of those elements $a \in G$ such that u and ua can be joined by an horizontal curve. We have that $\Phi(u)$ is a Lie subgroup of G, whose connected component of the identity is $\Phi^0(u)$, and we have isomorphisms $\Phi(u) \simeq \Phi(x)$ and $\Phi(u)^0 \simeq \Phi(x)^0$.

If $x, y \in M$ belong to the same symplectic leaf then the holonomy groups $\Phi(x)$ and $\Phi(y)$ are isomorphic. This is because if $u, v \in P$ are points such that, for some $a \in G$, there exists an horizontal curve connecting ua and v, then $\Phi(v) = Ad(a^{-1})\Phi(u)$, so $\Phi(u)$ and $\Phi(v)$ are conjugate in G. However, if $x, y \in M$ belong to different leaves the holonomy groups $\Phi(x)$ and $\Phi(y)$ will be, in general, non-isomorphic.

The holonomy groups can be given an infinitesimal description as in the Ambrose-Singer holonomy theorem. Suppose that $\gamma \in T^*_xM$ satisfies

$\#\gamma = 0$. If $u \in p^{-1}(x)$ we set

$$\Lambda(\gamma)_u \equiv \Lambda_j(\gamma)_x,$$

whenever $s_j(x) = u$. It follows from the transformation rule (1.23) that $\Lambda(\gamma)$ is well defined. Denoting by $P(u_0)$ the set of points $u \in P$ that can be joined to u_0 by a piecewise smooth horizontal curve, we have:

Theorem 1.5.2. *(Holonomy Theorem) Let Γ be a contravariant connection in $P(M,G)$, $u_0 \in P$. The Lie algebra of the holonomy group $\Phi(u_0) \subset G$ is the subalgebra of $\mathfrak{g}$ spanned by all elements of the form $\{\Xi(\alpha,\beta)_u, \Lambda(\gamma)_u\}$, where $u \in P(u_0)$ and $\alpha, \beta, \gamma \in T^*_{p(u)}M$, with $\#\gamma = 0$.*

Proof. Let $\mathfrak{g}(u_0)$ denote the subalgebra of $\mathfrak{g}$ spanned by all elements of the form $\{\Xi(\alpha,\beta)_u, \Lambda(\gamma)_u\}$, where $u \in P(u_0)$ and $\alpha, \beta, \gamma \in T^*_{p(u)}M$, with $\#\gamma = 0$. We claim that the generalized distribution $u \mapsto \mathcal{D}_u$ in $P(M,G)$, where

$$\mathcal{D}_u = \left\{ h(u,\alpha) + \sigma(A)_u : \ \alpha \in T^*_{p(u)}M, A \in \mathfrak{g}(u_0) \right\},$$

is integrable and that $P(u_0)$ is the integral leaf through u_0.

Assuming that this is the case the proposition follows, for we have for any $A \in \mathfrak{g}$

$$\begin{aligned} A \in \mathrm{Lie}(\Phi(u_0)) \quad &\Longleftrightarrow \quad a_t = \exp(tA) \in \Phi(u_0), \forall t \\ &\Longleftrightarrow \quad u_0 a_t \in p^{-1}(p(u_0)) \cap P(u_0), \forall t \\ &\Longleftrightarrow \quad \sigma(A)_{u_0} \in G_{u_0} \cap T_{u_0}P(u_0) = G_{u_0} \cap \mathcal{D}_{u_0} \\ &\Longleftrightarrow \quad A \in \mathfrak{g}(u_0), \end{aligned}$$

since $h(\alpha)_{u_0} + \sigma(A)_{u_0}$ is vertical iff $\#\alpha = 0$, and in this case $h(\alpha) = -\sigma(\Lambda(\alpha)_{u_0})$.

To prove the claim we observe that the smooth distribution $u \mapsto \mathcal{D}_u$ is integrable becouse it is involutive (use Lemma 1.4.4) and rank invariant. Let $G_{\mathcal{H}}$ be the group of diffeomorphism generated by the horizontal vector fields $h(\alpha)$. A theorem of Sussmann [9], shows that the $G_{\mathcal{H}}$-invariant distribution $\tilde{\mathcal{H}}$ generated by $\mathcal{H}$ is integrable and that $P(u_0)$ is a leaf through u_0 of $\tilde{\mathcal{H}}$. Moreover, $\tilde{\mathcal{H}}$ is the smallest integrable distribution containing $\mathcal{H}$. Therefore, the claim will follow if we can show that $\mathcal{D}_u = \tilde{\mathcal{H}}$.

On one hand, $\mathcal{D}$ is involutive and $\mathcal{H} \subset \mathcal{D}$, so we must have $\tilde{\mathcal{H}}_u \subset \mathcal{D}_u$. On the other hand, we have:

(a) $\tilde{\mathcal{H}}$ contains all horizontal vector fields since they also belong to $\mathcal{H}$;

(b) $\tilde{\mathcal{H}}$ contains all vector fields $\sigma(A)$ for $A = \Lambda(\gamma)_u$, with $\#\gamma = 0$, since in this case we have that $\Lambda(\gamma)_u = -h(\gamma)_u$ is horizontal;

(c) $\tilde{\mathcal{H}}$ contains all vector fields $\sigma(A)$ for $A = \Xi(\alpha,\beta)_u$, since by Proposition 1.4.3 $\sigma(A)$ is sum of an horizontal vector field and the Lie bracket of two horizontal vector fields;

We conclude that $\mathcal{D}_u \subset \tilde{\mathcal{H}}_u$ and the claim follows. q.e.d.

Note that the presence of the extra term $\Lambda(\gamma)$ implies that a connection can be flat and have non-discrete holonomy.

1.6 Mappings of connections

Recall that a homomorphism $\phi : P(M,G) \to P'(M',G')$ of principal bundles is a mapping of the total spaces $\phi : P \to P'$ such that $\phi(ua) = \phi(u)\varphi(a)$, $u \in P$, $a \in G$, where $\varphi : G \to G'$ is a Lie group homomorphism. We also have an induced map between the base spaces, denoted here by the same letter: $\phi : M \to M'$. If this map is a diffeomorphism and $s_j : U_j \to P$ is a local section of $P(M,G)$ then $s'_j : \phi(U_j) \to P'$ defined by $s'_j = \phi \circ s_j \circ \phi^{-1}$ is a local section of $P'(M',G')$.

Proposition 1.6.1. *Let M and M' be Poisson manifolds and $\phi : P(M,G) \to P'(M',G')$ a homomorphism such that the induced map $\phi : M \to M'$ is a Poisson isomorphism. Given a contravariant connection Γ in $P(M,G)$ there is a unique contravariant connection Γ' in $P'(M',G')$ such that ϕ maps horizontal subspaces of Γ to horizontal subspaces of Γ'. The connection vector fields and the curvature bivector fields of Γ and Γ' are related by:*

$$\Lambda'_j(\alpha) = \varphi_*\Lambda_j(\phi^*\alpha), \qquad \Xi'_j(\alpha,\beta) = \varphi_*\Xi_j(\phi^*\alpha,\phi^*\beta), \qquad \alpha,\beta \in \Omega^1(U'_j). \tag{1.34}$$

If $u \in P$ and $u' = \phi(u) \in P'$, then $\varphi : G \to G'$ maps the holonomy groups $\Phi(u)$ (resp. $\Phi^0(u)$) onto $\Phi(u')$ (resp. $\Phi^0(u')$).

Proof. To define the connection Γ', given $u' \in P'$ we choose $u \in P$ and $a' \in G'$ such that $u' = \phi(u)a'$, and set

$$h'(u',\alpha') = (R_{a'} \circ \phi)_* h(u, \phi^*\alpha').$$

One checks that this definition is independent of the choice of u and a'.

If $b' \in G$, then

$$\begin{aligned} h'(u'b', \alpha') &= (R_{a'b'} \circ \phi)_* h(u, \phi^* \alpha') = R_{b'*}(R_{a'} \circ \phi)_* h(u, \phi^* \alpha') \\ &= R_{b'*} h'(u', \alpha'), \end{aligned}$$

hence Γ' is invariant. By invariance, we can now assume $\phi(u) = u'$, and we have:

$$\begin{aligned} p'_* h'(u', \alpha') &= p'_* \phi_* h(u, \phi^* \alpha') \\ &= \phi_* p_* h(u, \phi^* \alpha') \\ &= \phi_* \# \phi^* \alpha' = \#' \alpha', \end{aligned}$$

since $\phi : M \to M'$ is a Poisson map. Therefore, Γ' is a contravariant connection.

From the relation

$$s'_{j*}(\#'\alpha) = \phi_* s_{j*}(\# \phi^* \alpha),$$

and the fact that the infinitesimal actions are related by

$$\sigma'(\varphi_* A) = \phi_* \sigma(A), \quad A \in \mathfrak{g},$$

we obtain formulas (1.34) for the connection vector fields. As for the curvature bivector fields we have:

$$\begin{aligned} \Xi'_j(\alpha, \beta) &= \delta' \Lambda'_j(\alpha, \beta) + \frac{1}{2}[\Lambda'_j, \Lambda'_j](\alpha, \beta) \\ &= \varphi_* \delta \Lambda_j(\phi^* \alpha, \phi^* \beta) + \frac{1}{2}[\varphi_* \Lambda_j, \varphi_* \Lambda_j](\phi^* \alpha, \phi^* \beta) \\ &= \varphi_* \delta \Lambda_j(\phi^* \alpha, \phi^* \beta) + \frac{1}{2}\varphi_*[\Lambda_j, \Lambda_j](\phi^* \alpha, \phi^* \beta) = \varphi_* \Xi_j(\phi^* \alpha, \phi^* \beta), \end{aligned}$$

for any forms $\alpha, \beta \in \Omega^1(U'_j)$.

Finally, if (γ', α') is a cotangent loop at $x' = p'(u')$ lying in the symplectic leaf through x', then $(\gamma, \alpha) = (\phi^{-1} \circ \gamma', \phi^* \alpha)$ is a cotangent loop at $x = p(u)$ lying in the symplectic leaf through x. Therefore, if $\tilde{\gamma}$ is a horizontal lift of (γ, α) then $\phi \circ \tilde{\gamma}$ is a horizontal lift of (γ', α') and so the holonomy groups must be related as stated. q.e.d.

In the situation of the previous proposition we say that ϕ maps the connection Γ to the connection Γ'. There are two important special cases to note:

a) if $P'(M',G')$ is a reduced sub-bundle of $P(M,G)$, so $M = M'$, $\phi : M \to M$ is the identity map, and $h : G \to G'$ is a monomorphism, we say the connection Γ' is *reducible* to the connection Γ;

b) if $P'(M',G') = P(M,G)$, $M = M'$ and $\Gamma = \Gamma'$ we say that the connection Γ is *invariant* by ϕ, or simply ϕ-invariant. This means precisely that:

$$h(\phi(u), \alpha) = \phi_* h(u, \phi^* \alpha), \qquad \forall (\phi(u), \alpha) \in p^* T^* M; \tag{1.35}$$

For a general Poisson map it is not possible to pullback or pushforward a contravariant connection, but there is still an obvious definition of *mapping of connections*.

1.7 Connections on fiber spaces

If G acts on the left on a manifold F we shall denote by

$$p_E : E(M, F, G, P) \to M$$

the fiber bundle associated with $P(M,G)$ with standard fiber F.

Given a connection Γ in $P(M,G)$ with associated horizontal lift $h : p^*T^*M \to TP$, we define the induced horizontal lift $h_E : p_E^* T^* M \to TE$ as follows: given $w \in E$ choose $(u, \xi) \in P \times F$ which is mapped to w, and set

$$h_E(w, \alpha) = \xi_* h(u, \alpha), \tag{1.36}$$

where we are identifying ξ with the map $P \to E$ which sends u to the equivalence class of (u, ξ). One can check easily that this definition does not depend on the choice of (u, ξ), so we obtain a well defined bundle map $h_E : p_E^* T^* M \to TE$ which makes the following diagram commute:

$$\begin{tikzcd} p_E^* T^* M \arrow[r, "h_E"] \arrow[d, "\widehat{p}_E"'] & TE \arrow[d, "p_{E*}"] \\ T^* M \arrow[r, "\#"'] & TM \end{tikzcd}$$

As before, we can define horizontal and vertical vectors in TE, horizontal lifts to E of curves lying on symplectic leaves of M, and parallel displacement of fibers of E. We shall call a cross section σ of E over an open set $U \subset M$ *parallel* if $\sigma_*(v)$ is horizontal for all tangent vectors $v \in T_U M$.

Theorem 1.7.1 (Reduction Theorem). *Let $P = P(M,G)$ be a principal fiber bundle over a Poisson manifold M with a contravariant connection Γ, and $H \subset G$ a closed subgroup. There exists a one to one correspondence between parallel cross sections $\sigma : M \to E(M, G/H, G, P)$ and sub-bundles $Q(M,H) \subset P(M,G)$ such that Γ is reducible to a connection Γ' in Q.*

Proof. Suppose we are given a parallel cross section $\sigma : M \to E(M, G/H, G, P)$. Let $\pi : P \to E$ be the natural projection. Then we define a sub-bundle $Q(M,H)$ by setting:

$$Q = \{u \in P : \pi(u) = \sigma(p(u))\}.$$

Given $u \in Q$ and $\alpha \in T^*_{p(u)}M$ let $(\gamma(t), \alpha(t))$ be a cotangent curve with $\gamma(0) = p(u)$ and $\alpha(0) = \alpha$. The horizontal lift $\tilde{\gamma}$ of this cotangent curve to P satisfies $\mu(\tilde{\gamma}(t)) = \sigma(\gamma(t))$, since σ is parallel. If follows that $h(u,\alpha) \in T_uQ$ for every $u \in Q$, so Γ is reducible to Q.

Conversely, suppose we are given a sub-bundle $Q(M,H)$ such that Γ is reducible to Q. Then we can define a section

$$\sigma : M \to E(M, G/H, G, P)$$

by setting $\sigma(x) = \pi(u)$, where $u \in Q$ is any point satisfying $p(u) = x$. If $\tilde{\gamma}(t)$ is an horizontal curve in P starting at $u \in Q$, then $\tilde{\gamma}(t) \in Q$ since Γ is reducible to Q. If $\gamma(t) = p(\tilde{\gamma}(t))$, it follows that $\mu(\tilde{\gamma}(t))$ is an horizontal lift of γ to E and that $\pi(\tilde{\gamma}(t)) = \sigma(\gamma(t))$, so σ is flat. q.e.d.

1.8 Relationship to ordinary connections

Let M be a symplectic manifold and Γ a contravariant connection on $P(M,G)$ with horizontal lift $h : p^*T^*M \to TP$. Then we have a bundle map $\tilde{h} : p^*TM \to TP$ defined by

$$\tilde{h}(u,v) = h(u, \#^{-1}v), \qquad (u,v) \in p^*TM.$$

This map is obviously G-invariant and makes the following diagram commute

$$\begin{tikzcd} p^*TM \arrow[r, "\tilde{h}"] \arrow[d, "\widehat{p}"'] & TP \arrow[d, "p_*"] \\ TM \arrow[r, "\mathrm{id}"'] & TM \end{tikzcd}$$

It follows that $\tilde{h}$ is the horizontal lift of a covariant connection on M. Let ω be the connection 1-form and let Ω be the curvature 2-form of this connection. Also, given trivialization isomorphisms $\{\psi_j\}$, inducing local sections $\{s_j\}$, set $\omega_j = s_j^*\omega$ and $\Omega_j = s_j^*\Omega$. Then it is clear from the definitions given above that the connection vector fields $\{\Lambda_j\}$ and the curvature bivector fields $\{\Xi_j\}$ are given by:

$$\Lambda_j = \#\omega_j, \qquad \Xi_j = \#\Omega_j. \tag{1.37}$$

For a general Poisson manifold with a contravariant connection Γ on $P(M,G)$ and horizontal lift $h : T^*M \to TP$, we say that Γ is *induced by a covariant connection* if

$$h(u,\alpha) = \tilde{h}(u,\#\alpha), \qquad (u,\alpha) \in p^*T^*M,$$

where $\tilde{h} : p^*TM \to TP$ is the horizontal lift of some covariant connection on M. Note that in this case the lift h satisfies:

$$\#\alpha = 0 \Longrightarrow h(u,\alpha) = 0, \qquad (u,\alpha) \in p^*T^*M. \tag{1.38}$$

This construction shows that there are always contravariant connections on any principal bundle $P(M,G)$ over a Poisson manifold M.

Not all connections satisfy property (1.38), so we set:

Definition 1.8.1. A contravariant connection Γ on a principal bundle $P(M,G)$ is called a $\mathcal{F}$-CONNECTION if its horizontal lift satisfies condition (1.38)

Assume we have a contravariant $\mathcal{F}$-connection Γ on $P(M,G)$. If $i : S \hookrightarrow M$ is a symplectic leaf, then on the pull-back bundle $\tilde{p} : i^*P \to S$ we have an induced connection Γ_S: on the total space $i^*P = \{(y,u) \in S \times P : i(y) = p(u)\}$ we define the horizontal lift $h_S : p_S^*T^*S \to T(i^*P)$ by setting

$$h_S((s,u),\alpha) = (p_*h(u,\beta), h(u,\beta)), \qquad (s,u) \in i^*P, (u,\alpha) \in p^*T^*M, \tag{1.39}$$

where $\beta \in T^*_{i(s)}M$ is such that $(d_si)^*\beta = \alpha$, and we are identifying $T(i^*P) = \{(v,w) \in TS \times TP : v = p_*w\}$. If $(d_si)^*\beta' = (d_si)^*\beta$, then $\#\beta' = \#\beta$, so we get the same result in (1.39) and so Γ is well defined. S being symplectic, the connection Γ_S is induced by a covariant connection on i^*P. Since the trivialization maps $\psi_j : p^{-1}(U_j) \to U_j \times G$ induce trivialization maps $\tilde{\psi}_j : \tilde{p}^{-1}(U_j \cap S) \to (U_j \cap S) \times G$ of the pull-back bundle $i^*P(M,G)$, writing $\tilde{s}_j(y) = \tilde{\psi}^{-1}(y,e)$ for the associated sections, we have:

Proposition 1.8.2. *Let Γ be an $\mathcal{F}$-connection in $P(M,G)$. If $x \in M$ and $i : S \hookrightarrow M$ is the symplectic leaf through x, denote by ω_S and Ω_S the connection 1-form and the curvature 2-form for the induced connection on $i^*P(M,G)$. Also, let $\omega_j = \tilde{s}_j^*\omega_S$ and $\Omega_j = \tilde{s}_j^*\Omega_S$. Then Λ_j and Ξ_j are i-related to $\#\omega_j$ and $\#\Omega_j$:*

$$i_*\#\omega_j = \Lambda_j, \qquad i_*\#\Omega_j = \Xi_j. \tag{1.40}$$

Therefore, a contravariant $\mathcal{F}$-connection in P can be thought of as a *family* of ordinary connections over the symplectic leaves of M. The (local) connection vector fields $\{\Lambda_j\}$ and the (local) curvature bivector fields $\{\Xi_j\}$ are obtained by gluing together the (local) connection vector fields $\{\#\omega_j\}$ and the (local) curvature bivector fields $\{\#\Omega_j\}$ of the connections on the symplectic leaves of M.

For an $\mathcal{F}$-connection, horizontal lifts of cotangent curves (γ, α) depend only on γ. Therefore, one has a well determined notion of horizontal lift of a curve lying on a symplectic leaf. It follows that for these connections, parallel displacement can also be defined by first reducing to the pull-back bundle over a symplectic leaf and then parallel displacing the fibers. Hence, the holonomy groups $\Phi(x)$ and $\Phi^0(x)$ coincide with the usual holonomy groups of the pull-back connection on the symplectic leaf S through x.

1.9 Flat connections

Let M be a Poisson manifold and $P(M,G) = M \times G$ the trivial principal bundle. The *canonical contravariant flat connection* in $P(M,G)$ is defined by taking as horizontal lift $h : p^*T^*M \to TP$ the map

$$h(u,\alpha) = (\#\alpha, 0), \qquad (u,\alpha) \in p^*T^*M$$

where we identify $TP = TM \times TG$. This connection is a $\mathcal{F}$-connection.

It is clear that a connection is the canonical flat connection iff it is reducible to the unique contravariant connection in $M \times e$, where $e \in G$ is the identity. For the canonical flat connection and the natural trivialization the connection vector field is $\Lambda = 0$, and so the canonical flat connection has zero curvature. Conversely, we have the following obvious proposition:

Proposition 1.9.1. *For an $\mathcal{F}$-connection Γ the following statements are equivalent:*

i) Γ *is flat;*

ii) every point has neighborhood U such that the induced connection in $P|_U$ is isomorphic to the canonical contravariant flat connection in $U \times G$;

iii) every point has neighborhood U such that there exists a parallel section $\sigma : U \to P$.

Moreover, a flat $\mathcal{F}$-connection has discrete holonomy.

If Γ is not an $\mathcal{F}$-connection the conclusions of the proposition, in general, do not hold.

2. Linear contravariant connections

2.1 Contravariant connections on a vector bundle

Let $P(M,G)$ be a principal bundle over a Poisson manifold M with a contravariant connection Γ. Suppose that G acts linearly on a vector space V, so on the associated vector bundle $E(M,V,G,P)$ we have the notion of parallel displacement of fibers along cotangent curves (γ,α) (see section 1.7).

Given a section ϕ of E defined along a cotangent curve (γ,α), we define the *contravariant derivative* $D_{(\gamma,\alpha)}\phi$ to be the section

$$D_{(\gamma,\alpha)}\phi(t) = \lim_{h\to 0}\frac{1}{h}\left[\tau_t^{t+h}(\phi(\gamma(t+h))) - \phi(\gamma(t))\right] \tag{2.1}$$

where $\tau_t^{t+h} : p_E^{-1}(\gamma(t+h)) \to p_E^{-1}(\gamma(t))$ denotes parallel transport of the fibers from $\gamma(t+h)$ to $\gamma(t)$ along the cotangent curve (γ,α).

Proposition 2.1.1. *Let ϕ and ψ be sections of E and f a function on M defined along γ. Then*

i) $D_{(\gamma,\alpha)}(\phi+\psi) = D_{(\gamma,\alpha)}\phi + D_{(\gamma,\alpha)}\psi$;

ii) $D_{(\gamma,\alpha)}(f\phi) = (f\circ\gamma)D_{(\gamma,\alpha)}\phi + \dot\gamma(f)(\phi\circ\gamma)$;

Proof. i) is obvious from the definition. On the other hand, we have

$$\tau_t^{t+h}(f(\gamma(t+h))\phi(\gamma(t+h))) = f(\gamma(t+h))\tau_t^{t+h}(\phi(\gamma(t+h))),$$

and ii) follows by the Leibniz rule. q.e.d.

Now let $\alpha \in T_x^*M$ be a covector and ϕ a cross section of E defined in a neighborhood of x. The contravariant derivative $D_\alpha\phi$ of ϕ in the direction of α is defined as follows: choose a cotangent curve $(\gamma(t), \alpha(t))$ defined for $t \in (-\varepsilon, \varepsilon)$, and such that $\gamma(0) = x$ and $\alpha(0) = \alpha$. Then we set:

$$D_\alpha\phi = D_{(\gamma,\alpha)}\phi(0). \tag{2.2}$$

It is easy to see that $D_\alpha\phi$ is independent of the choice of cotangent curve. Clearly, a cross section ϕ of E defined on an open set $U \subset M$ is flat iff $D_\alpha\phi = 0$ for all $\alpha \in T_xM$, $x \in M$.

Proposition 2.1.2. *Let $\alpha, \beta \in T_x^*M$, ϕ and ψ cross sections of E defined in a neighborhood U of x. Then*

i) $D_{\alpha+\beta}\phi = D_\alpha\phi + D_\beta\phi$;

ii) $D_\alpha(\phi + \psi) = D_\alpha\phi + D_\alpha\psi$;

iii) $D_{c\alpha} = cD_\alpha\phi$, *for any scalar* c;

iv) $D_\alpha(f\phi) = f(x)D_\alpha\phi + \#\alpha(f)\phi(x)$, *for any function* $f \in C^\infty(U)$;

Proof. iii) is obvious, while ii) and iv) follow from Proposition 2.1.1. To prove i) observe that any section ϕ of E, defined in a open set U, can be identified with a function $F : p^{-1}(U) \to V$ by letting

$$F(u) = u^{-1}(\phi(p(u))), \qquad u \in p^{-1}(U),$$

where we view $u \in P$ as a linear isomorphism $u : V \to p_E^{-1}(u)$. Then, as in the covariant case, we find

$$D_\alpha\phi = u(h(u, \alpha) \cdot F).$$

From this expression for the contravariant derivative, i) follows immediately. q.e.d.

Now let $\alpha \in \Omega^1(M)$ be a 1-form and ϕ a section of E. We define the contravariant derivative $D_\alpha\phi$ to be the section of E given by:

$$D_\alpha\phi(x) = D_{\alpha_x}\phi. \tag{2.3}$$

Proposition 2.1.3. *Let $\alpha, \beta \in \Omega^1(M)$, ϕ and ψ cross sections of E, and $f \in C^\infty(M)$. Then* *i)* $D_{\alpha+\beta}\phi = D_\alpha\phi + D_\beta\phi$;

ii) $D_\alpha(\phi + \psi) = D_\alpha\phi + D_\alpha\psi$;

iii) $D_{f\alpha} = fD_\alpha\phi$;

iv) $D_\alpha(f\phi) = fD_\alpha\phi + \#\alpha(f)\phi$;

Proof. From Proposition 2.1.2 we obtain immediately that i)-iv) hold. q.e.d.

It is also true that the contravariant derivative uniquely determines the connection. The proof of the following proposition is similar to the covariant case and so it will be omitted.

Proposition 2.1.4. *Suppose for each 1-form $\alpha \in \Omega^1(M)$ there is a linear operator D_α acting on sections of E and satisfying i)-iv) of Proposition 2.1.3. Then there exists a unique contravariant connection Γ on the associated principal bundle $P(M, G)$ whose induced contravariant derivative on E is D.*

In the case where the contravariant connection is induced by a covariant connection, the contravariant derivative D and the covariant derivative ∇ are related by

$$D_\alpha = \nabla_{\#\alpha}. \tag{2.4}$$

On the other hand, $\mathcal{F}$-connections can be characterized by the condition:

$$\#\alpha = 0 \implies D_\alpha = 0, \qquad \forall \alpha \in T^*(M). \tag{2.5}$$

Moreover, by Proposition 1.8.2, for an $\mathcal{F}$-connection, on each symplectic leaf $i : S \hookrightarrow M$ there is a covariant connection on the pullback bundle i^*P, inducing a covariant derivative ∇ on i^*E, with the following property: if ψ is any cross section of E, then

$$i^* D_\alpha \psi = \nabla_{\# i^*\alpha} i^*\psi, \tag{2.6}$$

where $i^*\psi$ denotes the section of the pullback bundle i^*E induced by ψ.

2.2 Linear contravariant connections

A *linear contravariant connection* is a contravariant connection on the coframe bundle $P = F^*(M)$ over M, so $G = GL(m)$ where $m = \dim M$. If $u = (\alpha_1, \dots, \alpha_m) \in F^*(M)$ is a coframe, we can view u as a linear isomorphism $u : (\mathbb{R}^m)^* \to T^*_{p(u)}M$ by setting

$$u(\xi)(v) = \xi(\alpha_1(v), \dots, \alpha_m(v)), \qquad v \in T_{p(u)}M,\ \xi \in (\mathbb{R}^m)^*.$$

We define the *canonical vector fields* θ_j on an open set U_j, with trivializing isomorphism $\psi_j : p^{-1}(U_j) \to U_j \times G$, and associated section $s_j(x) = \psi_j^{-1}(x, e)$, to be the $(\mathbb{R}^m)^*$-valued vector fields defined by

$$\theta_j(\alpha)_x = s_j(x)^{-1}(\alpha), \qquad x \in U_j. \tag{2.7}$$

These allows us to define the *torsion bivector fields* Θ_j to be the $(\mathbb{R}^m)^*$-valued bivector fields given by

$$\Theta_j(\alpha, \beta) = \delta\theta_j(\alpha, \beta) + \Lambda_j(\alpha) \cdot \theta(\beta) - \Lambda_j(\beta) \cdot \theta_j(\alpha). \tag{2.8}$$

Proposition 2.2.1. *The canonical vector fields and the torsion bivector fields of a linear contravariant connection are related by*

$$\theta_k = \psi_{jk}^{-1} \cdot \theta_j, \tag{2.9}$$

$$\Theta_k = \psi_{jk}^{-1} \cdot \Theta_j. \tag{2.10}$$

Moreover, they satisfy the Bianchi identity

$$\delta\Theta_j(\alpha, \beta, \gamma) = \bigodot_{\alpha,\beta,\gamma} \delta\Lambda_j(\alpha, \beta) \cdot \theta_j(\gamma) - \bigodot_{\alpha,\beta,\gamma} \Lambda_j(\alpha) \cdot \delta\theta_j(\beta, \gamma). \tag{2.11}$$

where the symbol $\bigodot$ denotes cyclic sum over the subscripts.

Proof. Relation (2.9) follows immediately from the definition of the canonical vector fields. To prove (2.10), we take the contravariant differential of (2.9):

$$\begin{aligned} \delta\theta_k(\alpha, \beta) =& \psi_{jk}^{-1} \cdot \delta\theta_j(\alpha, \beta) - \psi_{jk}^{-1}\delta\psi_{jk}(\alpha)\psi_{jk}^{-1} \cdot \theta_j(\beta) \\ &+ \psi_{jk}^{-1}\delta\psi_{jk}(\beta)\psi_{jk}^{-1} \cdot \theta_j(\alpha). \end{aligned}$$

From the transformation rule (1.23) for the connection vector fields, we find

$$\Lambda_k(\alpha) \cdot \theta_k(\beta) = \psi_{jk}^{-1}\Lambda_j(\alpha) \cdot \theta_j(\beta) + \psi_{jk}^{-1}\delta\psi_{jk}(\alpha)\psi_{jk}^{-1} \cdot \theta_j(\beta).$$

Therefore, we compute:

$$\begin{aligned}\Theta_k(\alpha,\beta) &= \delta\theta_k(\alpha,\beta) + \Lambda_k(\alpha)\cdot\theta(\beta) - \Lambda_k(\beta)\theta_j(\alpha)\\ &= \psi_{jk}^{-1}\delta\theta_j(\alpha,\beta) + \psi_{jk}^{-1}\Lambda_j(\alpha)\cdot\theta_j(\beta) - \psi_{jk}^{-1}\Lambda_j(\beta)\cdot\theta_j(\alpha)\\ &= \psi_{jk}^{-1}\cdot\Theta_j(\alpha,\beta).\end{aligned}$$

The Bianchi identity follows from taking the contravariant differential of (2.8). q.e.d.

For the standard contragradient action of $G = GL(m)$ on $F = (\mathbb{R}^m)^*$, the bundle associated with the coframe bundle $P = F^*(M)$ is the cotangent bundle $T^*M = E(M,F,G,P)$. Sections of $T^*(M)$ are just differential 1-forms and so the contravariant derivative associates to each 1-form α a linear operator $D_\alpha : \Omega^1(M) \to \Omega^1(M)$ such that:

$$D_{f_1\alpha_1+f_2\alpha_2} = f_1 D_{\alpha_1} + f_1 D_{\alpha_1}, \quad \text{for all } f_i \in C^\infty(M),\ \alpha_i \in \Omega^1(M), \tag{2.12}$$

$$D_\alpha(f\beta) = fD_\alpha\beta + \#\alpha(f)\beta, \quad \text{for all } f \in C^\infty(M),\ \alpha,\beta \in \Omega^1(M). \tag{2.13}$$

One can also consider other associated vector bundles to $F^*(M)$ which lead, just us in the covariant case, to contravariant derivatives of any tensor fields over M. For example, if X is a vector field, then $D_\alpha X$ is the contravariant derivative of X along the 1-form α. It is completely characterized by the relation

$$\langle D_\alpha X, \beta\rangle = \#\alpha(\langle X,\beta\rangle) - \langle X, D_\alpha\beta\rangle\,, \tag{2.14}$$

which holds for every 1-form $\beta \in \Omega^1(M)$. One has similar formulas for the contravariant derivative of any tensor field on M.

Local coordinate expressions for linear contravariant connections can be obtained in a way similar to the covariant case. Let $(x^1,\dots,x^m)$ be local coordinates on a neighborhood U in M. Then we define Christoffel symbols Γ_k^{ij} by

$$D_{dx^i}dx^j = \Gamma_k^{ij}dx^k. \tag{2.15}$$

It is easy to see that under a change of coordinates these symbols transform according to

$$\tilde{\Gamma}_n^{lm} = \frac{\partial y^l}{\partial x^i}\frac{\partial y^m}{\partial x^j}\frac{\partial x^k}{\partial y^n}\Gamma_k^{ij} + \frac{\partial y^l}{\partial x^i}\frac{\partial^2 y^m}{\partial x^j\partial x^k}\frac{\partial x^j}{\partial y^n}\pi^{ik}, \tag{2.16}$$

where π^{ik} are the components of the Poisson tensor. Conversely, given a family of symbols that transform according to this rule under a change of coordinates, we obtain a well defined contravariant derivative/connection on M.

Using these symbols, it is easy to get the local coordinates expressions for the contravariant derivatives: given a 1-form $\alpha = \alpha_i dx^i$ and a tensor field K, of type (r,s), with components $K^{i_1\dots i_r}_{j_1\dots j_s}$, we have

$$\begin{aligned}(D_\alpha K)^{i_1\dots i_r}_{j_1\dots j_s} =& \pi^{kl}\alpha_k \frac{\partial K^{i_1\dots i_r}_{j_1\dots j_s}}{\partial x^l} - \sum_{a=1}^{r}\left(\Gamma^{ki_a}_{l}\alpha_k K^{i_1\dots l\dots i_r}_{j_1\dots j_s}\right)\\ &+\sum_{b=1}^{s}\left(\Gamma^{kl}_{j_b}\alpha_k K^{i_1\dots i_r}_{j_1\dots l\dots j_s}\right).\end{aligned} \tag{2.17}$$

Given a tensor field K of type (r,s) we shall write, as in the covariant case, DK for the tensor field of type $(r+1,s)$ such that

$$(DK)^{i_1\dots i_r k}_{j_1\dots j_s} = (D_{dx^k}K)^{i_1\dots i_r}_{j_1\dots j_s}. \tag{2.18}$$

A tensor field K on M is *parallel* iff $DK = 0$.

2.3 Curvature and torsion tensor fields

For a linear contravariant connection on a Poisson manifold M we define the *torsion tensor field* T and the *curvature tensor field* R, respectively, to be the tensor fields of types $(2,1)$ and $(3,1)$ given by

$$T(\alpha,\beta) = s_j(x)(\Theta_j(\alpha,\beta)), \tag{2.19}$$

$$R(\alpha,\beta)\gamma = s_j(x)\left[\Xi^*_j(\alpha,\beta)\cdot s_j^{-1}(x)(\gamma)\right]. \tag{2.20}$$

where $x \in U_j$, $\alpha,\beta,\gamma \in T^*_x(M)$, and we are denoting by $\Xi^*_j(\alpha,\beta)$ the endomorphism of $\mathfrak{gl}(m)$ dual to $\Xi_j(\alpha,\beta)$. Note that if $x \in U_j \cap U_k$ and $s_k(x) = \psi_{jk}(x)s_k(x)$ we obtain the same values in formulas (2.19) and (2.20), so these really define tensor fields on all of M. These tensor fields can be easily expressed in terms of contravariant derivatives:

Proposition 2.3.1. *In terms of contravariant differentiation, the torsion T and the curvature R can be expressed as follows:*

$$T(\alpha,\beta) = D_\alpha\beta - D_\beta\alpha - [\alpha,\beta], \tag{2.21}$$

$$R(\alpha,\beta)\gamma = D_\alpha D_\beta\gamma - D_\beta D_\alpha\gamma - D_{[\alpha,\beta]}\gamma. \tag{2.22}$$

Moreover, the Bianchi identities (2.11) and (1.28) can also be expressed as

$$\bigodot_{\alpha,\beta,\gamma} \left(D_\alpha R(\beta,\gamma) + R(T(\alpha,\beta),\gamma)\right) = 0, \tag{2.23}$$

$$\bigodot_{\alpha,\beta,\gamma} \left(R(\alpha,\beta)\gamma - T(T(\alpha,\beta),\gamma) - D_\alpha T(\beta,\gamma)\right) = 0. \tag{2.24}$$

From formulas (2.21) and (2.22), we obtain immediately the following local coordinates expressions for the torsion and curvature tensor fields:

$$T_k^{ij} = \Gamma_k^{ij} - \Gamma_k^{ji} - \frac{\partial \pi^{ij}}{\partial x^k}, \tag{2.25}$$

$$R_l^{ijk} = \Gamma_l^{ir}\Gamma_r^{jk} - \Gamma_l^{jr}\Gamma_r^{ik} + \pi^{ir}\frac{\partial \Gamma_l^{jk}}{\partial x^r} - \pi^{jr}\frac{\partial \Gamma_l^{ik}}{\partial x^r} - \frac{\partial \pi^{ij}}{\partial x^r}\Gamma_l^{rk}. \tag{2.26}$$

Remark 2.3.2. Expressions (2.20) and (2.22) remain valid for any contravariant connection on a vector bundle E provided we replace γ by a section of E. In this case Bianchi's identity (1.28) can be expressed as

$$\bigodot_{\alpha_1,\alpha_2,\alpha_3} D_{\alpha_1}(R(\alpha_2,\alpha_3)) - \bigodot_{\alpha_1,\alpha_2,\alpha_3} R([\alpha_1,\alpha_2],\alpha_3) = 0.$$

If it happens that the contravariant connection is related to some covariant connection by:

$$\# D_\alpha \beta = \nabla_{\#\alpha} \# \beta,$$

(e. g., if D is induced by a covariant connection and Π is parallel, so $D_\alpha = \nabla_{\#\alpha}$ and $D\Pi = 0$) the torsion and curvature tensor fields are transformed by the musical homomorphism to the usual torsion and tensor fields of ∇:

$$T^\nabla(\#\alpha,\#\beta) = \# T^D(\alpha,\beta), \qquad R^\nabla(\#\alpha,\#\beta)\#\gamma = \# R^D(\alpha,\beta)\gamma.$$

2.4 Geodesics

For contravariant connections parallel transport can only be defined along curves lying in symplectic leaves of M. The same restriction applies to geodesics:

Definition 2.4.1. Let $(\gamma(t), \alpha(t))$ be a cotangent curve on M. We say that (γ, α) is a GEODESIC if:

$$(D_\alpha \alpha)_{\gamma(t)} = 0. \tag{2.27}$$

In local coordinates, a curve

$$(\gamma(t), \alpha(t)) = (x^1(t), \dots, x^m(t), \alpha_1(t), \dots, \alpha_m(t))$$

is a geodesic iff it satisfies the following system of ode's

$$\begin{cases} \frac{dx^i(t)}{dt} = \pi^{ji}(x^1(t), \dots, x^m(t))\alpha_j(t), \\ \\ \frac{d\alpha_i(t)}{dt} = -\Gamma_i^{jk}((x^1(t), \dots, x^m(t))\alpha_j \alpha_k. \end{cases} \qquad (i = 1, \dots, m) \tag{2.28}$$

From this we have:

Proposition 2.4.2. *Let M be a Poisson manifold, with a contravariant connection Γ, and $x_0 \in M$. Given $\alpha_{x_0} \in T^*_{x_0}M$, there is a unique maximal geodesic $t \mapsto (\gamma(t), \alpha(t))$, starting at $x_0 \in M$, with $\alpha(0) = \alpha_{x_0}$.*

Proof. Choose a systems of coordinates $(x^1, \dots, x^m)$ centered at x_0. By standard uniqueness and existence results for ode's, system (2.28) has a unique solution such that $(x^1(0), \dots, x^m(0), \alpha_1(0), \dots, \alpha_m(0)) = (0, \dots, 0, \alpha_{x_{0,1}}, \dots, \alpha_{x_{0,m}})$. q.e.d.

The geodesic given by this proposition is called the geodesic through x_0 with cotangent vector α_{x_0}. Note that if S is the symplectic leaf through x_0 and $v \in T_{x_0}S$ is a vector tangent to S, there can be several geodesics with this tangent vector at x_0. However, for an $\mathcal{F}$-connection geodesics are uniquely determined by tangent vectors and coincide with the geodesics of the covariant connection induced on S.

The following result is the analogue of a well known result in affine geometry:

Proposition 2.4.3. *Let Γ be a contravariant connection on M. There exists a unique contravariant connection on M with the same geodesics and zero torsion.*

Proof. Choose local coordinates on M so D has symbols Γ_k^{ij}, and consider the set of functions

$$^*\Gamma_k^{ij} = \frac{1}{2}\left(\Gamma_k^{ij} + \Gamma_k^{ji} + \frac{\partial \pi^{ij}}{\partial x^k}\right) \tag{2.29}$$

One checks that if Γ_k^{ij} and $\tilde{\Gamma}_n^{lm}$ are related by the transformation law (2.16), then ${}^*\Gamma_k^{ij}$ and ${}^*\tilde{\Gamma}_n^{lm}$ are also related by the same transformation law. It follows that we have a well defined contravariant connection D^* on M. From the local coordinate expressions for the torsion (2.25) and the geodesics (2.28), we see that D^* has zero torsion and the same geodesics as D.

For uniqueness, let D and D^* be two connections with the same geodesics and torsion 0. We let

$$S(\alpha,\beta) = D_\alpha\beta - D^*_\alpha\beta, \qquad \alpha,\beta\in\Omega^1(M). \tag{2.30}$$

Then S is C^∞-linear, so it is a tensor. Since the connections have 0 torsion, we have:

$$\begin{aligned} S(\alpha,\beta) - S(\beta,\alpha) &= (D_\alpha\beta - D_\beta\alpha) - (D^*_\alpha\beta - D^*_\beta\alpha) \\ &= [\alpha,\beta] - [\alpha,\beta] = 0. \end{aligned} \tag{2.31}$$

so S is a symmetric tensor. Now if $\alpha_p \in T^*_pM$, we can choose the geodesic (for D and D^*) with cotangent vector α_p and associated 1-form α along γ. We have

$$S(\alpha_p,\alpha_p) = D_\alpha\alpha - D^*_\alpha\alpha = 0, \tag{2.32}$$

so $S=0$ and $D=D^*$. q.e.d.

2.5 Poisson connections

Linear contravariant connections for which the Poisson tensor is parallel play an important role. Recall that a covariant connection for which the Poisson tensor is parallel exists iff the Poisson manifold has constant rank (see e. g. [10], thm. 2.20). On the other hand, for contravariant connections a simple argument involving a partition of unity shows that we have:

Proposition 2.5.1. *Every Poisson manifold has a linear contravariant connection with contravariant derivative D such that $D\Pi = 0$.*

Proof. Let U_a be a domain of a chart $(x^1,\dots,x^m)$. On U_a, the contravariant connection $D^{(a)}$ with symbols

$$\Gamma_k^{ij} = \frac{\partial\pi^{ij}}{\partial x^k}$$

satisfies $D^{(a)}\Pi = 0$. If we take an open cover of M by such chart domains and if $\sum_a \phi^{(a)} = 1$ is partition of unity subordinate to this cover, then $D = \sum_a \phi^{(a)} D^{(a)}$ is a connection on M for which Π is parallel. q.e.d.

We shall call a contravariant connection on M such that the Poisson tensor Π is parallel a POISSON CONNECTION. In the symplectic case, these coincide with the symplectic connections.

Suppose a Poisson connection has vanishing torsion. Since $D\Pi = 0$, we have $D\# = \#D$, and from $T = 0$ we conclude that for $\alpha, \beta \in \Omega^1(M)$

$$\begin{aligned}\#\alpha = 0 \Longrightarrow \#D_\alpha\beta &= \#D_\beta\alpha + \#[\alpha,\beta] \\ &= D_\beta\#\alpha + [\#\alpha,\#\beta] = 0.\end{aligned}$$

This should be compared with condition (2.5) for an $\mathcal{F}$-connection.

On the other hand, if we do have a Poisson $\mathcal{F}$-connection with vanishing torsion, then for each symplectic leaf $i : S \hookrightarrow M$ there exists a unique covariant symplectic connection ∇^S on S, such that

$$i^* D_\alpha \beta = \nabla^S_{\# i^*\alpha} i^*\beta, \quad \alpha, \beta \in \Omega^1(M).$$

As we pointed out above, a non-regular Poisson manifold does not admit covariant connections for which the Poisson tensor is parallel. Therefore, in general, it is not possible to glue together the covariant connections ∇^S to get a connection on M. As the following example shows, the family formed by these connections will develop singularities at points where the rank drops.

Example 2.5.2. Consider a 2-dimensional non-abelian Lie algebra $\mathfrak{g}$ and choose a basis $\{\omega_1, \omega_2\}$ such that:

$$[\omega_1, \omega_2] = \omega_1.$$

On $\mathfrak{g}^*$ we take the Lie-Poisson bracket which relative to the coordinates (x^1, x^2) defined by the dual basis satisfies $\left\{x^1, x^2\right\} = x^1$. Now consider the contravariant connection on $\mathfrak{g}^*$ defined by:

$$D_{dx^1} dx^1 = D_{dx^2} dx^2 = D_{dx^2} dx^1 = 0, \qquad D_{dx^1} dx^2 = dx^2.$$

One checks easily that D has zero torsion and $D\pi = 0$. On the other hand there is no globally defined covariant connection ∇ on $\mathfrak{g}^*$ such that $D_\alpha = \nabla_{\#\alpha}$. In fact, if such a connection existed, then denoting by Γ^k_{ij} its Christoffel symbols, we should have

$$\Gamma^{ij}_k = \pi^{il}\Gamma^j_{lk},$$

where Γ^{ij}_k are the symbols of D. Taking $i = k = 1$, $j = 2$, this would give

$$1 = x^1 \Gamma^2_{21},$$

which is impossible.

Note that formally we obtain the solution $\Gamma^2_{21} = \frac{1}{x^1}$, so there exists a singular connection with singular set $x^1 = 0$. This is precisely the set of points where the rank drops from 2 to 0.

3. Poisson holonomy

3.1 Holonomy of a symplectic leaf

For a regular foliation the topological behaviour close to a given leaf is controlled by the holonomy of the leaf. For a singular foliation, as is the case of the symplectic foliation of a Poisson manifold, there is in general no such notion of holonomy (see, however, [2] where holonomy is defined for transversely stable leaves). It turns out that in the case of a Poisson manifold it is still possible to introduce a notion of holonomy which also reflects the Poisson geometry of nearby leaves. In this theory of holonomy, contravariant connections play a significant role.

Let M be a Poisson manifold and let $i : S \hookrightarrow M$ be a symplectic leaf of M. Denote by $\nu(S) = T_S M/TS$ the normal bundle to S and by $p : \nu(S) \to S$ the natural projection. By the tubular neighborhood theorem, there exists a smooth immersion $\tilde{i} : \nu(S) \to M$ satisfying the following properties:

i) $\tilde{i}|_Z = i$, where Z is the zero section of $\nu(S)$;

ii) $\tilde{i}$ maps the fibers of $\nu(S)$ transversely to the symplectic foliation of M;

Assume that we have fixed such an immersion. Each fiber $F_x = p^{-1}(x)$ determines a splitting $T_x\nu(S) = T_xS \oplus T_xF_x$, so we have a decomposition:

(3.1)
$$T^*_x\nu(S) = T^*_xS \oplus T^*_uF_x, \qquad \text{where } (T_xF_x)^0 \simeq T^*_xS, \quad (T_xS)^0 \simeq T^*_xF_x.$$

Note that $T_xS = \mathrm{Im}\#_x = \#(T_xF_x)^0$. For each $u \in F_x$ we have an analogous splitting $T_u\nu(S) = \#(T_uF_x)^0 \oplus T_uF_x$, so there is also a decomposition:

$$T_u^*\nu(S) = (T_uF_x)^0 \oplus T_u^*F_x, \qquad \text{where } T_u^*F_x \simeq (\#(T_uF_x)^0)^0. \tag{3.2}$$

Each such immersion induces a unique Poisson structure on the total space $\nu(S)$ such that $\tilde{i} : \nu(S) \to M$ is a Poisson map. Also, on each fiber $F_x = p^{-1}(x)$ there is an induced *transverse Poisson structure* $\Pi_x^{\perp}$: The corresponding bundle map $\#^{\perp} : T^*F_x \to TF_x$ is defined as the composed map

$$T^*F_x \xrightarrow{q_x^*} T_{F_x}^*\nu(S) \xrightarrow{\#} T_{F_x}\nu(S) \xrightarrow{q_x} TF_x,$$

where $q_x : T_{F_x}\nu(S) \to TF_x$ is the bundle projection from the restricted tangent bundle $T_{F_x}\nu(S)$ onto TF_x associated with the decomposition (3.2).

Now let $\alpha \in T_xM$. We decompose α according to (3.1):

$$\alpha = \alpha^{||} + \alpha^{\perp}, \text{where } \alpha^{||} \in (T_xF_x)^0 \simeq T_x^*S, \quad \alpha^{\perp} \in (T_xS)^0 \simeq T_x^*F_x.$$

Since F_x is a linear space, there is a natural identification $T_x^*F_x \simeq T_u^*F_x$, and we denote by $\tilde{\alpha}_u^{\perp} \in T_u^*F_x \simeq (\#(T_uF_x)^0)^0$ the element corresponding to $\alpha^{\perp}$. On the other hand, the composition of the musical isomorphism $\#$ with the differential of the projection $p : \nu(S) \to S$ induces an isomorphism between the annihilator $(T_uF_x)^0$ and T_xS, so we also have an isomorphism $(T_uF_x)^0 \simeq T_x^*S$. If we denote by $\tilde{\alpha}_u^{||} \in (T_uF_x)^0$ the element corresponding to $\alpha^{||}$ under this isomorphism, we have $p_*\#\tilde{\alpha}^{||} = \#\alpha$.

Given a covector $\alpha \in T_x^*M$ we shall define its *horizontal lift* to $\nu(S)$ by

$$h(u, \alpha) = \#\tilde{\alpha}_u^{||} + \#^{\perp}\tilde{\alpha}_u^{\perp} \in T_u\nu(S).$$

By construction, we have property (CI)* of a contravariant connection

$$p_*h(u, \alpha) = \#\alpha, \quad u \in p^{-1}(x),$$

so this horizontal lift defines a kind of generalized contravariant connection in $\nu(S)$. Note that it depends both on the immersion and on the Poisson tensor.

Let $(\gamma(t), \alpha(t))$, $t \in [0,1]$, be a cotangent curve in the symplectic leaf S starting at $x = \gamma(0)$. If $u \in \nu(S)|_x$ is a point in the fiber over x, there exists an $\varepsilon > 0$ and a horizontal curve $\tilde{\gamma}(t)$ in $\nu(S)$, defined for $t \in [0, \varepsilon)$, which satisfies:

$$\begin{cases} \frac{d}{dt}\tilde{\gamma}(t) = h(\tilde{\gamma}(t), \alpha(t)), & t \in [0, \varepsilon), \\ \tilde{\gamma}(0) = u. \end{cases}$$

Moreover, we can choose a neighborhood U_γ of $0 \in \nu(S)|_x$, such that for each $u \in U_\gamma$ the lift $\tilde{\gamma}(t)$ with initial point u is defined for all $t \in [0,1]$.

If $(\gamma(t), \alpha(t))$ is a cotangent loop based at $x \in S$ then this lift gives, by passing from initial to end point, a diffeomorphism $H_S(\gamma, \alpha)$ of U_γ into another neighborhood V_γ of $0 \in \nu(S)|_x$, with the property that 0 is mapped to 0. One extends the definition of H_S for piecewise smooth cotangent loops in the obvious way.

Denote by $\mathfrak{Aut}(F_x)$ the group of germs at 0 of Poisson automorphisms of F_x which map 0 to 0.

Proposition 3.1.1. *Let $(\gamma, \alpha), (\gamma', \alpha')$ be cotangent loops based at $x \in S$, then:*

i) $H_S(\gamma, \alpha)$ is an element of $\mathfrak{Aut}(F_x)$;

ii) $H_S((\gamma, \alpha) \cdot (\gamma', \alpha')) = H_S(\gamma, \alpha) \circ H_S(\gamma', \alpha')$, where the dot denotes concatenation of cotangent loops.

Proof. Let $(\gamma(t), \alpha(t))$ be a cotangent curve in S. For each t, we have a trivialization of $p : \nu(S) \to S$ in a neighborhood of $\gamma(t)$ such that $p(x, y) = x$. If $\alpha(t) = \sum a(t) dx|_{\gamma(t)} + b(t) dy|_{\gamma(t)}$ we consider the 1-form with constant coefficients $\alpha_t = \sum a(t) dx + b(t) dy$. The lift of its restriction to S defines the time-dependent vector field:

$$X_t = \#\tilde{\alpha}_t^{||} + \#^{\perp} \tilde{\alpha}_t^{\perp},$$

where

$$\tilde{\alpha}_t^{||} \in (TF_{\gamma(t)})^0, \ \tilde{\alpha}_t^{\perp} \in T^*F_{\gamma(t)} \simeq (\#(TF_{\gamma(t)})^0)^0.$$

For each t, the transverse component $\tilde{\alpha}_t^{\perp}$ is a closed 1-form in $F_{\gamma(t)}$.

The lifts $\tilde{\gamma}$ of γ are the integral curves of the vector field X_t. We claim that the flow ϕ^t of this vector field preserves the transverse Poisson structure $\Pi^{\perp}$

$$(\phi^{-t})_* \Pi^{\perp}_{\phi^t(u)} = \Pi^{\perp}_u, \tag{3.3}$$

so (i) follows. Part (ii) also follows since we have just shown that we can take $H_S(\gamma, \alpha)$ as the time-1 map of some flow.

To prove (3.3) we observe that

$$\frac{d}{dt} (\phi^{-t})_* \Pi^{\perp}_{\phi^t(u)} = (\phi^{-t})_* \left[\frac{d}{dh} (\phi^{-h})_* \Pi^{\perp}_{\phi^h(\phi^t(u))} \right]_{h=0},$$

and we use the following lemma:

Lemma 3.1.2. *If* $\alpha_1, \alpha_2 \in T_u^* F_x \simeq (\#(T_u F_x)^0)^0$ *then*

$$\left[\frac{d}{dh}(\phi^{-h})_* \Pi^{\perp}_{\phi^h(u)}\right]_{h=0} (\alpha_1, \alpha_2) = (\mathcal{L}_{X_t}\Pi)_u(\alpha_1, \alpha_2)$$

Now we have

$$\begin{aligned}\mathcal{L}_{X_t}\Pi(\alpha_1, \alpha_2) &= \mathcal{L}_{\#\tilde{\alpha}_t^{\|}}\Pi(\alpha_1, \alpha_2) + \mathcal{L}_{\#^{\perp}\tilde{\alpha}_t^{\perp}}\Pi(\alpha_1, \alpha_2) \\ &= \mathcal{L}_{\#\tilde{\alpha}_t^{\|}}\Pi(\alpha_1, \alpha_2) + \mathcal{L}_{\#^{\perp}\tilde{\alpha}_t^{\perp}}\Pi^{\perp}(\alpha_1, \alpha_2)\end{aligned}$$

The transverse component vanishes since $\tilde{\alpha}_t^{\perp}$ is a closed form in the fiber, for each t. For the parallel component we write $\tilde{\alpha}_t^{\|} = \sum_i a_i dx^i$, and we compute

$$\mathcal{L}_{\#\tilde{\alpha}_t^{\|}}\Pi = \sum_i \left(a_i \mathcal{L}_{\#dx^i}\Pi + \#da_i \wedge \#dx^i\right).$$

But $dx^i \in (TF_x)^0$ and since $\alpha_1, \alpha_2 \in (\#(T_u F_x)^0)^0$ we conclude that

$$\mathcal{L}_{\#\tilde{\alpha}_t^{\|}}\Pi(\alpha_1, \alpha_2) = \sum_i a_i \mathcal{L}_{\#dx^i}\Pi(\alpha_1, \alpha_2) = 0,$$

so the parallel component also vanishes.

It remains to prove Lemma 3.1.2. We note that for any $\alpha \in T_u^* F_x$ we have $q^*_{\phi^h(u)}(\phi^{-h})^*\alpha - (\phi^{-h})^* q_u^*\alpha \in (TF_{p(\phi^h(u))})^0$. Using this remark we find:

$$\begin{aligned}&\left[\frac{d}{dh}(\phi^{-h})_* \Pi^{\perp}_{\phi^h(u)}\right]_{h=0} (\alpha_1, \alpha_2) \\ &= \lim_{h\to 0}\frac{1}{h}\left[\Pi_{\phi^h(u)}(q^*_{\phi^h(u)}(\phi^{-h})^*\alpha_1, q^*_{\phi^h(u)}(\phi^{-h})^*\alpha_2) - \Pi_u(q_u^*\alpha_1, q_u^*\alpha_2)\right] \\ &= \Pi_{\phi^h(u)}\left[\Pi_{\phi^h(u)}((\phi^{-h})^* q_u^*\alpha_1, (\phi^{-h})^* q_u^*\alpha_2) - \Pi_u(q_u^*\alpha_1, q_u^*\alpha_2)\right] \\ &= (\mathcal{L}_{X_t}\Pi)_u(q_u^*\alpha_1, q_u^*\alpha_2),\end{aligned}$$

so the lemma follows. q.e.d.

Denoting by $\Omega_*(S, x)$ the group of piecewise smooth cotangent loops, we see that we have a map $H_S : \Omega_*(S, x) \to \mathfrak{Aut}(F_x)$, which will be called the *Poisson holonomy* of the leaf S. Note that the Poisson holonomy map depends on the immersion $\tilde{i} : \nu(S) \to M$, but two different immersions lead to conjugate homomorphisms.

Example 3.1.3. Let S be a regular leaf of a Poisson manifold M. In decomposition 3.2 we can identify $(T_uF_x)^0 \simeq T_u^*S_u$ and $(T_uS_u)^0 \simeq T_u^*F_u$, where S_u is the symplectic leaf through u. It follows that the horizontal lift $h(u,\alpha)$ is the unique tangent vector in T_uS_u which projects to $\#\alpha$. We conclude that for a regular leaf the Poisson holonomy coincides with the usual holonomy.

Example 3.1.4. Let $\mathfrak{g}$ be some finite dimensional Lie algebra and consider on $M = \mathfrak{g}^*$ the canonical linear Poisson bracket. For the singular leaf $S = \{0\}$ we have $\nu(S) \simeq \mathfrak{g}^*$ with $p(u) \equiv 0$ and the decomposition 3.2 collapses. Given a covector $\alpha \in T_0^*\mathfrak{g}^* = \mathfrak{g}$ we find $h(u,\alpha) = \#_u\alpha = \operatorname{ad}^*\alpha \cdot u$. It follows that for a constant cotangent loop $(0,\alpha)$ in S we have $H_S(0,\alpha) = \operatorname{Ad}^*(\exp(\alpha))$, which of course is a Poisson automorphism of $F_0 \simeq \mathfrak{g}^*$.

3.2 Reduced Poisson holonomy

As example 3.1.4 shows, Poisson holonomy is not a homotopy invariant. Following the construction given in [4] for the linear case, we can give a notion of *reduced Poisson holonomy* which is homotopy invariant.

For a Poisson manifold M let us denote by $\operatorname{Aut}(M)$ the group of Poisson diffeomorphisms of M, and by $\operatorname{Aut}^0(M)$ its connected component of the identity: given $\phi \in \operatorname{Aut}^0(M)$ there exists a smooth family $\phi_t \in \operatorname{Aut}(M)$, $t \in [0,1]$, such that $\phi_0 = \mathrm{id}$, $\phi_1 = \phi$, and ϕ_t is generated by a time-dependent vector field:

$$\frac{d\phi_t}{dt} = X_t \circ \phi_t.$$

The vector field X_t is an infinitesimal Poisson automorphism:

$$\mathcal{L}_{X_t}\Pi = 0.$$

We shall say that ϕ is a *inner Poisson automorphism* or a *hamiltonian automorphism* if there exists a smooth family of hamiltonian functions $h_t : M \to \mathbb{R}$ such that $X_t = X_{h_t} = \#dh_t$. The set $\operatorname{Inn}(M) \subset \operatorname{Aut}(M)$ of inner Poisson automorphisms is a normal subgroup, and we define the group of *outer Poisson automorphisms* of M to be the quotient $\operatorname{Out}(M) = \operatorname{Aut}(M)/\operatorname{Inn}(M)$

Recall that for a symplectic leaf S we denote by $\mathfrak{Aut}(F_x)$ the group of germs at 0 of Poisson automorphisms of F_x which map 0 to 0. We shall also denote by $\mathfrak{Out}(F_x)$ the corresponding group of germs of outer Poisson automorphisms.

Proposition 3.2.1. *Let S be a symplectic leaf of M, with Poisson holonomy $H_S : \Omega_*(S,x) \to \mathfrak{Aut}(F_x)$. If (γ_1,α_1) and (γ_2,α_2) are cotangent loops with $\gamma_1 \sim \gamma_2$ homotopic then $H_S(\gamma_1,\alpha_1)$ and $H_S(\gamma_2,\alpha_2)$ represent the same equivalence class in $\mathfrak{Out}(F_x)$.*

Proof. Since any piecewise smooth path $\gamma \subset S$ can be made into a cotangent path, by property (ii) in Proposition 3.1.1 it is enough to show that for every $x \in S$ there exists a neighborhood U of x in S such that if $\gamma(t) \subset U$ is a piecewise smooth loop based at x and $\alpha(t) \in T^*M$ is a piecewise smooth family with $\#\alpha = \dot{\gamma}$ then $H_S(\gamma,\alpha) \in \mathrm{Inn}\,(F_x)$.

To see this we use the same notation as in the proof of Proposition 3.1.1. In a trivializing neighborhood U of $p : \nu(S) \to S$ containing x, we can decompose the vector field X_t as:

$$X_t = \#\tilde{\alpha}_t^{||} + \#^{\perp}\tilde{\alpha}_t^{\perp},$$

where

$$\tilde{\alpha}_t^{||} \in (TF_{\gamma(t)})^0,\ \tilde{\alpha}_t^{\perp} \in T^*F_{\gamma(t)} \simeq (\#(TF_{\gamma(t)})^0)^0.$$

For each t, the transverse component $\tilde{\alpha}_t^{\perp}$ can be taken to be a closed 1-form in $F_{\gamma(t)}$. It is clear that the parallel component $\#\tilde{\alpha}_t^{||}$ has no effect on the holonomy. Hence we can assume that $S = \{x\}$, $F_x = M$, γ is a constant path and $\tilde{\alpha}_t^{\perp} = \tilde{\alpha}_t$, so

$$X_t = \#\tilde{\alpha}_t = \#dh_t,$$

for some function h_t defined in a neighborhood of x. Since $H_S(\gamma,\alpha)$ is the time-1 flow of this hamiltonian vector field we conclude that $H_S(\gamma,\alpha) \in \mathrm{Inn}\,(F_x)$. q.e.d.

Given a loop γ in S we shall denote by $\bar{H}_S(\gamma) \in \mathfrak{Out}(F_x)$ the equivalence class of $H_S(\gamma,\alpha)$ for some piece-wise smooth family $\alpha(t)$ with $\#\alpha(t) = \gamma(t)$. The map $\bar{H}_S : \Omega(S,x) \to \mathfrak{Out}(F_x)$ will be called the *reduced Poisson holonomy homomorphism* of S. This maps extends to continuous loops and, by a standard argument, it induces a homomorphism $\bar{H}_S : \pi_1(S,x) \to \mathfrak{Out}(F_x)$ where $\pi_1(S,x)$ is the fundamental group (the use of the same letter to denote both these maps should not be the cause of any confusion).

3.3 Stability

The reduced Poisson holonomy of a leaf carries information on the behaviour of the Poisson structure in a neighborhood of the leaf. The simplest result in this direction can be obtained as follows: let us call S *transversely stable* if the transverse Poisson manifold N is stable near $S \cap N$, i. e., if N has arbitrarily small neighborhoods of $N \cap S$ which are invariant under all hamiltonian automorphisms.

Theorem 3.3.1. *(Local Stability I) Let S be a compact, transversely stable leaf, with finite reduced holonomy. Then S is stable, i. e., S has arbitrarily small neighborhoods which are invariant under all hamiltonian automorphisms. Moreover, each symplectic leaf of M near S is a bundle over S whose fiber is a finite union of symplectic leaves of the transverse Poisson structure.*

Proof. Assume first that S has trivial reduced holonomy. We fix an embedding $\tilde{i} : \nu(S) \to M$ as above and a base point $x_0 \in S$. Also, we choose a Riemannian metric on S.

By compactness of S, there exists a number $c > 0$ such that every point $x \in S$ can be connected to x_0 by a smooth cotangent path of length $< c$. For some inner product on $\nu(S)|_{x_0}$, let D_ε be the disk of radius ε centered at 0. For each $\varepsilon > 0$, there exists a neighborhood $U \subset D_\varepsilon$ such that:

i) for any piecewise-smooth cotangent path in S, starting at x_0, with length $\leq 2c$ and for any $u \in U$, there exists a lifting with initial point u;

ii) the lifting of any cotangent loop based at x_0 with initial point $u \in U$ has end point in U;

iii) U is invariant under all hamiltonian automorphisms;

In fact, let $(\gamma_1, \alpha_1), \ldots, (\gamma_k, \alpha_k)$ be cotangent loops such that $\gamma_1, \ldots, \gamma_k$ are generators of $\pi_1(S, x_0)$, and let ϕ_i be Poisson diffeomorphisms which represent the germs $H_S(\gamma_i, \alpha_i)$. Since the reduced holonomy is trivial, there is a neighborhood U' of 0 in $F_{x_0} = \nu(S)|_{x_0}$ such that $U' \subset \mathrm{domain}(\phi_1) \cap \cdots \cap \mathrm{domain}(\phi_k)$, and $\phi_i|U' \in \mathrm{Inn}\,(F_{x_0})$, for all i. Since S is transversely stable, we can choose a smaller neighborhood $U \subset U'$ invariant under all hamiltonian automorphisms.

Given $x \in S$ and a cotangent path (γ, α) connecting x_0 to x, let us denote by $\sigma_{(\gamma,\alpha)} : U \to F_x$ the diffeomorphism defined by lifting. It follows from i) and ii) above that if (γ', α') is a cotangent path homotopic to (γ, α) then $\sigma_{(\gamma,\alpha)}(U) = \sigma_{(\gamma',\alpha')}(U)$. It follows from iii) that $\sigma_{(\gamma,\alpha)}(U)$ is also invariant under all hamiltonian automorphisms.

Let V be a neighborhood of S in M. There exists $\varepsilon(x) > 0$ such that for the corresponding $U_x \subset D_{\varepsilon(x)}$ we have $\sigma_{(\gamma,\alpha)}(U_x) \subset V \cap F_x$. By compactness of S, we can choose $\varepsilon > 0$ (independent of $x \in S$) such that for the corresponding $U \subset D_\varepsilon$ we have

$$\sigma_{(\gamma,\alpha)}(U) \subset V \cap F_x$$

Set

$$V_0 = \bigcup_{(\gamma,\alpha)} \sigma_{(\gamma,\alpha)}(U).$$

Then $V_0 \subset V$ is a open neighborhood of S which is invariant under all hamiltonian automorphisms of M.

If $u, u' \in V_0$ are two points in the same symplectic leaf such that $p(u) = p(u') = x$, then there is a path $\tilde{\gamma}$ in this symplectic leaf connecting these two points. It follows from the decomposition (3.2) that there exists a cotangent loop (γ, α) in S such that $\tilde{\gamma}$ is a horizontal lift of this loop. Thus u' is the image of u by $H_S(\gamma, \alpha)$ which is a hamiltonian automorphism of $V_0 \cap F_x$. Therefore, u and u' lie in the same symplectic leaf of $V_0 \cap F_x$. We conclude that each symplectic leaf of M near S is a bundle over S whose fiber is a symplectic leaf of the transverse Poisson structure.

Assume now that S has finite reduced Poisson holonomy. We let $q : \tilde{S} \to S$ be a finite covering space such that $q_*\pi_1(\tilde{S}) = \text{Ker } \bar{H}_S \subset \pi_1(S)$. If we embed $\nu(S)$ into M as above, and let $\nu(\tilde{S})$ be the pull back bundle of $\nu(S)$ over $\tilde{S}$, we have a unique Poisson structure in $\nu(\tilde{S})$ such that the natural map $\nu(\tilde{S}) \to \nu(S)$ is a Poisson map. Moreover, the reduced Poisson holonomy of $\nu(\tilde{S})$ along $\tilde{S}$ is trivial, so we can apply the above argument to $\nu(\tilde{S})$ and the theorem follows. q.e.d.

Remark 3.3.2. If a leaf S is transversely stable and $x \in S$, let N denote a stable neighborhood of F_x. For each cotangent path (γ, α), the Poisson holonomy $H_S(\gamma, \alpha)$ induces a homeomorphism of the orbit space of N, for the transverse Poisson structure, mapping zero to zero. If (γ_1, α_1) and (γ_2, α_2) are cotangent loops such that $H_S(\gamma_1, \alpha_1)$ and $H_S(\gamma_2, \alpha_2)$ represent the same class in $\mathfrak{Out}(F_x)$, then they induce the same germ of homeomorphism of the orbit space mapping zero to zero. In [2] holonomy of a general, transversely stable, foliation is defined using germs of homeomorphisms of the orbit space, which in the case of a Poisson manifold coincide with these ones.

3.4 Strict Poisson holonomy

Another problem raised by the local splitting theorem and related to stability is whether one has a global splitting of an entire neighborhood of a leaf S. Note that if a neighborhood V of S has a Poisson splitting $S \times N$ then projection to the first factor is a Poisson map. This motivates the

Definition 3.4.1. Let M be a Poisson manifold and $i : S \hookrightarrow M$ a symplectic leaf of M. A POISSON TUBULAR NEIGHBORHOOD of S is a smooth immersion $\tilde{i} : \nu(S) \to M$ satisfying:

i) $\tilde{i}|_Z = i$, where Z is the zero section of $\nu(S)$;

ii) $\tilde{i}$ maps the fibers of $\nu(S)$ transversely to the symplectic foliation of M;

iii) For the Poisson structure on $\nu(S)$ induced from $\tilde{i}$, the canonical projection $p : \nu(S) \to S$ is a Poisson map;

Suppose S admits a Poisson tubular neighborhood. Then the regular distribution $\#(\mathrm{Ker}\ p_*)^0$ is integrable and S, identified with the zero section, is an integral leaf of this distribution. Hence, we can consider the holonomy of S (in the usual sense) as a leaf of the corresponding foliation. We call this the *strict Poisson holonomy* of S, and we denote by $\check{H}_S : \Omega(S, x) \to \mathfrak{Diff}(F_x)$ the associated holonomy map, where $\mathfrak{Diff}(F_x)$ denotes the group of germs of diffeomorphisms of F_x which map 0 to 0. Strict Poisson holonomy is related to reduced Poisson holonomy as follows.

Proposition 3.4.2. *Assume S admits a Poisson tubular neighborhood. The map $\check{H}_S : \Omega(S, x) \to \mathfrak{Diff}(F_x)$ has image inside $\mathfrak{Aut}(F_x)$ and the following diagram commutes:*

$$\begin{tikzcd}
\Omega(S,x) \arrow[r, "\check{H}_S"] \arrow[dr, "\bar{H}_S"'] & \mathfrak{Aut}(F_x) \arrow[d] \\
& \mathfrak{Out}(F_x)
\end{tikzcd}$$

Proof. Fix a Poisson tubular neighborhood $p : \nu(S) \to S$ and consider the generalized connection in $\nu(S)$ defined by the distribution $\#(\mathrm{Ker}\ p_*)^0$. Given a loop $\gamma(t)$ in S there exists a family of closed forms $\alpha_t^S \in \Omega^1(S)$ such that $\#\alpha_t^S(\gamma(t)) = \dot{\gamma}(t)$. The horizontal lifts of this loop are integral curves of the time-dependent vector field

$$\check{X}_t = \#p^*\alpha_t^S.$$

Since $dp^*\alpha_t^S = p^*d\alpha_t^S = 0$, this vector field is an infinitesimal Poisson automorphism. We conclude that the holonomy maps $\check{H}_S(\gamma)$ are Poisson automorphisms.

Moreover, in the notation of the proof of Proposition 3.2.1, we have $\check{X}_t = \#\alpha_t^{||}$. It follows that if (γ, α) is a cotangent loop in M then $H_S(\gamma, \alpha)$ and $\check{H}_S(\gamma)$ represent the same class in $\mathrm{Out}\,(F_x)$. q.e.d.

We can now state and prove the following splitting result:

Theorem 3.4.3. *(Local Stability II) Suppose $i : S \hookrightarrow M$ is a compact symplectic leaf of a Poisson manifold M which admits a Poisson tubular neighborhood. Assume further that S has finite strict Poisson holonomy and let $q : \tilde{S} \to S$ be the finite covering corresponding to Ker $\check{H}_S \subset \pi_1(S, x)$. Then there is a neighborhood V of S and a finite covering Poisson map $\phi : \tilde{S} \times N \to V$, where N is a transverse Poisson manifold to S. If S is transversely stable, then we can choose N and V to be stable neighborhoods.*

Proof. By a standard homotopy lifting argument, as in the end of the proof of Theorem 3.3.1, it is enough to consider the special case where the holonomy is trivial. We must then show that there is a neighborhood V of S and a Poisson diffeomorphism $\phi : S \times N \to V$, where N is a transverse Poisson manifold to S.

Again, we fix an embedding $\tilde{i} : \nu(S) \to M$ as above and a base point $x_0 \in S$. Also, we choose a Riemannian metric on S. By compactness of S, there exists a number $c > 0$ such that every point $x \in S$ can be connected to x_0 by a smooth cotangent path of length $< c$. For some inner product on $\nu(S)|_{x_0}$, let D_ε be the disk of radius ε centered at 0. There exists an $\varepsilon > 0$ such that: for any piecewise-smooth cotangent path in S, starting at x_0, with length $\leq 2c$ and for any $u \in D_\varepsilon$, there exists a lifting with initial point u. Moreover, by shrinking ε if necessary, we can assume that the lifting of any cotangent loop based at x_0 with initial point u also ends at u. In fact, let $(\gamma_1, \alpha_1), \ldots, (\gamma_k, \alpha_k)$ be cotangent loops such that $\gamma_1, \ldots, \gamma_k$ are generators of $\pi_1(S, x_0)$, and let ϕ_i be Poisson diffeomorphisms which represent the germs $\check{H}_S(\gamma_i, \alpha_i)$. Then, since the holonomy is trivial by assumption, there is a neighborhood

U of 0 in $\nu(S)|_{x_0}$ such that $U \subset \text{domain}(\phi_1) \cap \cdots \cap \text{domain}(\phi_k)$, and $\phi_i|U$ =identity, for all i. We need only to choose ε such that $D_\varepsilon \subset U$.

For each $u \in D_\varepsilon$ we define a map $\sigma_u : S \to M$ as follows: let $x \in S$ and connect x to x_0 by a cotangent path (γ, α) of length $< c$. Let $\tilde{\gamma}$ be the unique lift of (γ, α) starting at u, and define $\sigma_u(x) = \tilde{\gamma}(1)$. This map is well defined because the holonomy is trivial. Also, σ_u is clearly a local embedding since $p \circ \sigma_u$ =identity on S. Since S is compact we conclude that σ_u is an embedding.

The map σ_u clearly depends smoothly on u, and since the holonomy is trivial, the map $u \mapsto \sigma_u(x)$, for a fixed x, is one-to-one. It follows that the map $\phi : S \times D_\varepsilon \to M$ given by $(x, u) \mapsto \sigma_u(x)$ is a diffeomorphism onto a neighborhood V of S.

By hypothesis, $p : \nu(S) \to S$ is a Poisson map. On the other hand, the composition $p \circ \phi : S \times D_\varepsilon \to S$ is just projection into the first factor, which is also a Poisson map. Then ϕ must also be a Poisson map.

Finally, if S is transversely stable, we can choose an open set $N \subset D_\varepsilon$ stable for the transverse structure, so $V = \phi(S \times N)$ is a stable neighborhood. q.e.d.

For simply connected leaves we obtain:

Corollary 3.4.4. *Let $i : S \hookrightarrow M$ be a compact, simply connected, symplectic leaf of a Poisson manifold M, which admits a Poisson tubular neighborhood. Then there is a neighborhood V of S and a Poisson diffeomorphism $\phi : S \times N \to V$, where N is a transverse Poisson manifold to S. If S is transversely stable, then we can choose N and V to be stable neigborhoods.*

One should note that, in general, a leaf does not have a Poisson tubular neighborhood, and so strict Poisson holonomy is not defined. In the following example we give a Poisson manifold M with a compact, simply connected, symplectic leaf S, which has no Poisson tubular neighborhood. In particular, M does not split as $S \times N$ in a neighborhood of S.

Example 3.4.5. First observe that $CP(n)$ is a coadjoint orbit of $U(n+1)$, since the standard action of $U(n+1)$ on $CP(n)$ is a transitive hamiltonian action. In fact, a theorem of Kostant says that, for a compact Lie group, all hamiltonian G-spaces on which G acts transitively are coadjoint orbits. The argument goes as follows: Let $\Phi : CP(n) \to \mathfrak{u}^*(n+1)$ be the (equivariant) moment map. Then $U(n+1)$ acts transitively on the image $Y = \Phi(CP(n))$, which therefore is a coadjoint orbit. In fact, $\Phi : CP(n) \to Y$ is a symplectomorphism and, since $CP(n)$ is compact, Φ is a covering map. However, every coadjoint orbit of a compact Lie group is simply connected (see [5], sect. 9.4), so this map is actually a diffeomorphism.

Consider in particular the case $n = 2$. We claim that $CP(2)$ is a symplectic leaf of $\mathfrak{u}^*(3)$ which has no Poisson tubular neighborhood. In fact, if $CP(2)$ had such a Poisson tubular neighborhood then it would have trivial strict Poisson holonomy and, by Theorem 3.4.3, its normal bundle $\nu(CP(n))$ would be trivial. But this is not the case, as can be seen from the following standard argument: the total Chern class of $CP(2)$ is $c = (1+a)^3 = 1+3a+3a^2$, where a is a generator of $H^2(CP(2), \mathbb{Z})$. The total Stiefel-Whitney class w of $CP(2)$ is the image of c by the canonical homomorphism $H^2(CP(2), \mathbb{Z}) \to H^2(CP(2), \mathbb{Z}_2)$ and hence is non-zero. The total Stiefel-Whitney class of the normal bundle $\nu(CP(2))$ is w^{-1}, which is non-trivial. We conclude that $\nu(CP(2))$ is non-trivial.

3.5 Linear Poisson holonomy

Let M be a Poisson manifold and $i : S \hookrightarrow M$ a symplectic leaf of M with Poisson holonomy $H_S : \Omega_*(S, x) \to \mathfrak{Aut}(F_x)$ (once a tubular neighborhood has been fixed).

On $T_0 F_x \simeq F_x$ we consider the Poisson bivector field Π^L which is the linear approximation at 0 to the Poisson bracket on F_x. Also, we denote by $\mathrm{Aut}\,(F_x)$ the set of linear Poisson automorphisms of (F_x, Π^L). There is a map $d : \mathfrak{Aut}(F_x) \to \mathrm{Aut}\,(F_x)$ which assigns to a germ of a Poisson diffeomorphism of $(F_x, \Pi^\perp)$, mapping zero to zero, its linear approximation.

Definition 3.5.1. The LINEAR POISSON HOLONOMY of the leaf S is the homomorphism $H_S^L \equiv dH_S : \Omega_*(S, x) \to \mathrm{Aut}\,(F_x)$.

One can check that this notion of linear Poisson holonomy is essentially the same as the one introduced in [4].

To define the *reduced linear Poisson holonomy* of the leaf S one can either show that the class of $H_S^L(\gamma, \alpha)$ in

$$\mathrm{Out}\,(F_x) = \mathrm{Aut}\,(F_x)/\mathrm{Inn}\,(F_x)$$

is homotopy invariant, or else take the composition

$$\bar{H}_S^L \equiv \bar{d}\bar{H}_S : \pi_1(S, x) \to \mathrm{Out}\,(F_x),$$

where $\bar{d} : \mathfrak{Out}(F_x) \to \text{Out}\,(F_x)$ is the natural map. Similarly, if S admits a Poisson tubular neighborhood, one can define the *strict linear Poisson holonomy* as the composition $\check{H}_S^L \equiv d\check{H}_S : \pi_1(S, x) \to GL(F_x)$.

One can give a differential operator formulation for linear Poisson holonomy similar to the Bott connection of ordinary foliation theory. Instead of working with the normal bundle $\nu(S) = T_S M/TS$ it is convenient to use the dual bundle $\nu^*(S)$, also called the conormal bundle. We have natural identifications

$$\nu^*(S) = (\text{Ker } \#)|_S = (TS)^0.$$

On $\nu^*(S)$ we have the following contravariant analogue of the Bott connection: Given a covector $\alpha \in T_S^* M$ and a section β of $\nu^*(S)$, take forms $\tilde{\alpha}, \tilde{\beta} \in \Omega^1(M)$ such that $\tilde{\alpha}_x = \alpha$, $\tilde{\beta}|_S = \beta$, and we set:

$$D_\alpha^S \beta \equiv [\tilde{\alpha}, \tilde{\beta}]|_S. \tag{3.4}$$

To check that this definition is independent of the extensions considered, we note that, by (1.3), it can also be written as

$$D_\alpha^S \beta = \mathcal{L}_{\#\tilde{\alpha}} \tilde{\beta}|_S. \tag{3.5}$$

Expression (3.4) also shows that $D_\alpha^S \beta$ is in the kernel of $\#$ and so is a section of $\nu^*(S)$. Therefore, D^S associates to each 1-form α on M along S a differential operator $D_\alpha : \Gamma(\nu^*(S)) \to \Gamma(\nu^*(S))$.

It is also easy to check that D^S satisfies the analogue of properties i)-iv) of Proposition 2.1.3. Note however that, in general, D^S *does not give* a contravariant connection in $\nu^*(S)$, since it is defined only for 1-forms in M along S. One can now define parallel transport of fibers of $\nu^*(S)$ along cotangent curves in S, and hence linear holonomy of D^S. The holonomy of D^S coincides with the linear Poisson holonomy introduced above.

It is convenient to consider the connections D^S all together, rather than leaf by leaf, so we set:

Definition 3.5.2. A linear contravariant connection D on M is called a BASIC CONNECTION if

i) D restricts to D^S on each leaf S, i. e., if $\alpha, \beta \in \Omega^1(M)$ and $\#\beta|_S = 0$ then
$$D_\alpha \beta|_S = D_\alpha^S \beta.$$

ii) D preserves the Poisson tensor, i. e.,

$$D\Pi = 0.$$

It is clear that one can also define linear Poisson holonomy starting with some basic connection. The holonomy of this basic connection determines maps of each cotangent space T_x^*M which map $\ker \#_x$ isomorphically into itself, and these are the linear Poisson holonomy maps.

Basic connections always exist:

Proposition 3.5.3. *Every Poisson manifold has basic connections. If D is a basic connection with curvature tensor R, and γ is a 1-form such that $\#\gamma|_S = 0$, then*

$$R(\alpha, \beta)\gamma|_S = 0.$$

Proof. Assume first that $M \simeq \mathbb{R}^m$, with coordinates $(x^1, \dots, x^m)$. We define a contravariant connection on M by setting

$$D_{dx^j}\beta = [dx^j, \beta].$$

Then, obviously, if S is a leaf of M and $\#\beta|_S = 0$ we have

$$D_{dx^j}\beta|_S = D^S_{dx^j}\beta.$$

It follows that for any 1-form α we have

$$D_\alpha\beta|_S = D^S_\alpha\beta.$$

Moreover, $D\Pi = 0$ so D is a basic connection.

For an arbitrary Poisson manifold M we choose an open cover $\{U^{(a)}\}$, with a partition of unity $\sum_a \phi_a = 1$ subordinate to this cover, and such that on each $U^{(a)}$ there is a basic connection $D^{(a)}$. Then $D = \sum_a \phi_a D^{(a)}$ is a basic connection.

If D is any basic connection and $\#\gamma|_S = 0$, we have $D_\alpha\gamma|_S = [\alpha, \gamma]|_S$ for any 1-form α, so expression (2.22) for the curvature tensor, gives

$$R(\alpha, \beta)\gamma|_S = [\alpha, [\beta, \gamma]]|_S - [\beta, [\alpha, \gamma]]|_S - [[\alpha, \beta], \gamma]|_S.$$

But the right hand side is zero, because of Jacobi identity. q.e.d.

Remark 3.5.4. Although the curvature of a basic connection vanishes along $\ker \#$, the holonomy along $\#$ need not be discrete (this is because of the presence of an extra term in the holonomy Theorem 1.5.2). Hence, in general, linear Poisson holonomy is not discrete and also not homotopy invariant (cf. example 3.1.4). However, if one can find a basic connection which is an $\mathcal{F}$-connection, then Poisson holonomy is discrete. Such is the case for a regular Poisson manifold, where (linear) Poisson holonomy coincides with standard (linear) holonomy.

The following result by now should be obvious.

Proposition 3.5.5. *Let M be a Poisson manifold, and S a symplectic leaf which admits a transverse measure μ invariant under the hamiltonian flow. Then, for every cotangent path (γ, α) in S*

$$\det H_S^L(\gamma, \alpha) = 1,$$

where the determinant is computed relative to μ.

This result also follows from a formula of Ginzburg and Golubev, proved in [4], which states that for *any* measure μ on M one has

$$\det H_S^L(\gamma, \alpha) = \exp\left(\int_{(\gamma,\alpha)} v_\mu\right), \tag{3.6}$$

where v_μ is the modular vector field of the measure μ and the determinant is computed relative to the measure induced by μ on the transverse fiber (see section 4.4). This formula shows that there is a strong relationship between the modular class and Poisson holonomy. In the next section we will introduce invariants of a Poisson manifold which generalize the modular class, and we will make this relationship more precise.

4. Characteristic classes

4.1 Poisson-Chern-Weil Homomorphism

The usual Chern-Weil theory for characteristic classes extend to contravariant connections, as was observed in [11]. We give here a short account since we shall need characteristic classes later in the section.

Consider a principal G-bundle $p : P \to M$ over a Poisson manifold, and choose some contravariant connection Γ on P. Given any symmetric, $\mathrm{Ad}\,(G)$-invariant, k-multilinear function

$$P : \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$$

we can define a $2k$-vector field $\lambda(\Gamma)(P)$ on M as follows. If U_j is a trivializing neighborhood, $x \in U_j$ and $\alpha_1, \dots, \alpha_{2k} \in T_x^*M$ then we set

$$\begin{aligned}(4.1)\qquad &\lambda(\Gamma)(P)(\alpha_1, \dots, \alpha_{2k}) \\ &\quad = \sum_{\sigma \in S_{2k}} (-1)^\sigma P(\Xi_j(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}), \dots, \Xi_j(\alpha_{\sigma(2k-1)}, \alpha_{\sigma(2k)})).\end{aligned}$$

By the transformation rule for the curvature bivector fields, this formula actually defines a 2k-vector field on the whole of M.

Proposition 4.1.1. *For any symmetric, invariant, k-multilinear function P, the $2k$-vector field $\lambda(\Gamma)(P)$ is closed:*

$$(4.2)\qquad \delta\lambda(\Gamma)(P) = 0.$$

Proof. We compute

$$\begin{aligned}\delta\lambda(\Gamma)(P) &= kP(\delta\Xi_j, \dots, \Xi_j) \\ &= kP(\delta\Xi_j + [\Lambda_j, \Xi_j], \dots, \Xi_j) = 0,\end{aligned}$$

where we have used first the linearity and symmetry of P, then the $\mathrm{Ad}\,(G)$-invariance of P, and last the Bianchi identity. q.e.d.

Therefore, to each invariant, symmetric, k-multilinear function $P \in I^k(G)$ we can associate a Poisson cohomology class $[\lambda(\Gamma)(P)] \in H^{2k}_\Pi(M)$, and in fact we have:

Proposition 4.1.2. *The cohomology class $[\lambda(\Gamma)(P)]$ is independent of the contravariant connection used to define it.*

Proof. Consider two contravariant connections Γ^0 and Γ^1 in P. Then we have a family of connections Γ^t with connection vector fields $\Lambda_j^t = t\Lambda_j^1 + (1-t)\Lambda_j^0$. We denote by Ξ_j^t its curvature bivector fields. Also, the difference $\Lambda_j^{1,0} = \Lambda_j^1 - \Lambda_j^0$ is a $\mathfrak{g}$-valued vector field. By the transformation rule (1.23), given $P \in I^k(G)$, we get a well defined $(2k-1)$-vector field $\lambda(\Gamma^1, \Gamma^0)(P)$ by setting

$$\begin{aligned}(4.3)\qquad &\lambda(\Gamma^1, \Gamma^0)(P)(\alpha_1, \dots, \alpha_{2k-1}) \\ &\quad = k \sum_{\sigma \in S_{2k-1}} (-1)^\sigma \int_0^1 P(\Lambda_j^{1,0}(\alpha_{\sigma(1)}), \Xi_j^t(\alpha_{\sigma(2)}, \alpha_{\sigma(3)}), \\ &\qquad\qquad \dots, \Xi_j^t(\alpha_{\sigma(2k-2)}, \alpha_{\sigma(2k-1)}))dt.\end{aligned}$$

We claim that

$$\delta\lambda(\Gamma^1, \Gamma^0) = \lambda(\Gamma^1)(P) - \lambda(\Gamma^0)(P), \tag{4.4}$$

so $[\lambda(\Gamma^1)(P)] = [\lambda(\Gamma^0)(P)]$.

To prove (4.4), we note that if we differentiate the structure equation (1.26) we obtain

$$\frac{d}{dt}\Xi_j^t = \delta\Lambda_j^{1,0} + [\Lambda_j^t, \Lambda_j^{1,0}]. \tag{4.5}$$

Hence, using Bianchi's identity, we have

$$\begin{aligned}
&k\delta\int_0^1 P(\Lambda_j^{1,0}, \Xi_j^t, \dots, \Xi_j^t)dt \\
&\quad = k\int_0^1 P(\delta\Lambda_j^{1,0}, \Xi_j^t, \dots, \Xi_j^t) \\
&\qquad + P(\Lambda_j^{1,0}, \delta\Xi_j^t, \dots, \Xi_j^t) + P(\Lambda_j^{1,0}, \Xi_j^t, \dots, \delta\Xi_j^t)dt \\
&\quad = k\int_0^1 P(\frac{d}{dt}\Xi_j^t - [\Lambda_j^t, \Lambda_j^{1,0}], \Xi_j^t, \dots, \Xi_j^t) \\
&\qquad - P(\Lambda_j^{1,0}, [\Lambda_j^t, \Xi_j^t], \dots, \Xi_j^t) \\
&\qquad - P(\Lambda_j^{1,0}, \Xi_j^t, \dots, [\Lambda_j^t, \Xi_j^t])dt \\
&\quad = k\int_0^1 P(\frac{d}{dt}\Xi_j^t, \Xi_j^t, \dots, \Xi_j^t)dt \\
&\quad = \int_0^1 \frac{d}{dt}P(\Xi_j^t, \Xi_j^t, \dots, \Xi_j^t)dt = P(\Xi_j^1, \dots, \Xi_j^1) \\
&\qquad - P(\Xi_j^0, \dots, \Xi_j^0).
\end{aligned}$$

q.e.d.

If we set

$$I^*(G) = \bigoplus_{k\geq 0} I^k(G),$$

the assignment $P \mapsto [\lambda(\Gamma)(P)]$ gives a map $I^*(G) \to H^*_\Pi(M)$. This map is in fact a ring homomorphism.

Proposition 4.1.3. *The following diagram commutes*

$$\begin{tikzcd}
I^*(G) \arrow[r] \arrow[dr] & H^*(M) \arrow[d, "\#"] \\
 & H^*_\Pi(M)
\end{tikzcd}$$

where on the top row we have the Chern-Weil homomorphism.

Proof. Choose a contravariant connection Γ in P which is induced by a covariant connection $\tilde{\Gamma}$. Given $P \in I^k(G)$, we have a closed $(2k)$-form $\lambda(\tilde{\Gamma})(P)$ defined by a formula analogous to (4.1), and which induces the Chern-Weil homomorphism $I^*(G) \to H^*(M)$. We check easily that

$$\#\lambda(\tilde{\Gamma})(P) = \lambda(\Gamma)(P),$$

so the proposition follows. q.e.d.

Recall that the ring $I^*(GL_q(\mathbb{R}))$ is generated by elements $P_k \in I^k(GL_q(\mathbb{R}))$ such that $P_k(A, \ldots, A) = \sigma_k(A)$, where $\{\sigma_1, \ldots, \sigma_q\}$ are the elementary symmetric functions defined by:

$$\det(\mu I - \frac{1}{2\pi}A) = \mu^q + \sigma_1(A)\mu^{q-1} + \cdots + \sigma_q(A).$$

Now consider a real vector bundle $p_E : E \to M$ over a Poisson manifold, with fiber $F \simeq \mathbb{R}^q$ and let $p : P \to M$ be the associated principal bundle with structure group $GL_q(\mathbb{R})$. Choosing a contravariant connection Γ on P one defines the *kth Poisson-Pontrjagin class* of E as

$$p_k(E, \Pi) = [\lambda(\Gamma)(P_{2k})] \in H^{4k}_{\Pi}(M).$$

As usual, one does not need to consider the classes for odd k since we have

$$[\lambda(\Gamma)(P_{2k-1})] = 0,$$

as can be seen by choosing a connection compatible with a riemannian metric. It is clear from Proposition 4.1.3 that

$$p_k(E, \Pi) = \#p_k(E).$$

where $p_k(E)$ are the standard Pontrjagin classes of E. Note also, that if $r = \operatorname{rank} M = \max_{x\in M}(\operatorname{rank}\Pi_x)$ we have $p_k(E, \Pi) = 0$ for $k > r/2$.

To compute these invariants one uses the contravariant derivative operator D on E, associated with the contravariant connection Γ, and proceeds as follows. For covectors $\alpha, \beta \in T_xM$, the curvature tensor R defines a linear map $R_{\alpha,\beta} = R(\alpha, \beta) : F_x \to F_x$ which satisfies $R_{\alpha,\beta} = -R_{\beta,\alpha}$, and so $(\alpha, \beta) \to R_{\alpha,\beta}$ can be considered as a $\mathfrak{gl}(E)$-valued bivector field. By fixing a basis of local sections, we have $F_x \simeq \mathbb{R}^q$ so we have $R_{\alpha,\beta} \in \mathfrak{gl}_q(\mathbb{R})$. (this matrix representation of $R_{\alpha,\beta}$ is defined only up to a change of basis in $\mathbb{R}^q$). Hence, if

$$P : \mathfrak{gl}_q(\mathbb{R}) \times \cdots \times \mathfrak{gl}_q(\mathbb{R}) \to \mathbb{R}$$

is a symmetric, k-multilinear function, $\mathrm{Ad}\,(GL_q(\mathbb{R}))$-invariant, we a have a $2k$-vector field $\lambda(R)(P)$ on M defined by

$$\begin{aligned}\lambda(R)(P)&(\alpha_1,\ldots,\alpha_{2k})\\&=\sum_{\sigma\in S_{2k}}(-1)^\sigma P(R_{\alpha_{\sigma(1),\sigma(2)}},\ldots,R_{\alpha_{\sigma(2k-1),\sigma(2k)}}).\end{aligned}\tag{4.6}$$

It is easy to see that $\lambda(\Gamma)(P)=\lambda(R)(P)$, so this gives a procedure to compute the Poisson-Chern-Weil homomorphism and the Poisson-Pontrjagin classes.

Similar considerations apply to other characteristic classes. One can define, e. g., the Poisson-Chern classes $c_k(E,\Pi)$ of a complex vector bundle E over a Poisson manifold, and they are just the images by $\#$ of the usual Chern classes of E.

The fact that all these classes arise as images by $\#$ of some known classes is perhaps a bit disappointing. However, we shall see below that one can define Poisson secondary characteristic classes which are intrinsic of Poisson geometry, and which do not arise as images by $\#$ of some de Rham cohomology classes.

4.2 Secondary characteristic classes

We shall now introduce secondary characteristic classes of a Poisson manifold. We will see that these classes give information on the topology, as well as, the geometry of the symplectic foliation. As in the theory of (regular) foliations, these classes appear when we compare two connections, each from a distinguished class.

On the Poisson manifold M, with $\dim M=m$, we consider the following data:

i) A basic connection Γ^1, with a contravariant derivative D^1;

ii) A linear contravariant connection Γ^0 induced by a riemannian connection, so $D^0_\alpha=\nabla^0_{\#\alpha}$ with $\nabla^0 g=0$ for some riemannian metric g;

Given an invariant, symmetric, k-multilinear function $P\in I^k(GL(m,\mathbb{R}))$ we consider the $(2k-1)$-vector field $\lambda(\Gamma^1,\Gamma^0)(P)$ given by (4.3).

Proposition 4.2.1. *If k is odd, $\lambda(\Gamma^1,\Gamma^0)(P)$ is a closed $(2k-1)$-vector field.*

Proof. According to (4.4) we have

$$\delta\lambda(\Gamma^1,\Gamma^0)(P)=\lambda(\Gamma^1)(P)-\lambda(\Gamma^0)(P).$$

and we claim that $\lambda(\Gamma^1)(P)=\lambda(\Gamma^0)(P)=0$ if k is odd.

The proof that $\lambda(\Gamma^0)(P)=0$ is standard: since there exists a metric such that $D^0g=0$ we can reduce the structure group of Γ^0 to $O(m,\mathbb{R})$, so the curvature bivector fields take their values in $\mathfrak{so}(m,\mathbb{R})$. But if $A\in\mathfrak{so}(m,\mathbb{R})$, we have $P_k(A)=0$ for any elementary symmetric function, since k is odd. Hence we obtain $\lambda(\Gamma^0)(P)=0$.

Consider now the connection Γ^1. Given $x\in M$ we choose local coordinates (x^j,y^k) around x as in the Weinstein splitting theorem:

$$\Pi=\sum_{i=1}^n\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^{i+n}}+\sum_{k,l}\phi_{kl}\frac{\partial}{\partial y^k}\wedge\frac{\partial}{\partial y^l},$$

where $\phi_{kl}(x)=0$. Since Γ^1 is a basic connection, we have:

$$\Pi(D^1_\alpha dx^i,dx^j)=-\Pi(dx^i,D^1_\alpha dx^j),\qquad R^1(\alpha,\beta)dy^k|_x=0.$$

It follows that $R^1(\alpha,\beta)_x$ is represented in the basis (dx^j,dy^k) by a matrix of the form:

$$\begin{pmatrix} B & 0\\ C & 0\end{pmatrix},\tag{4.7}$$

with B a symplectic matrix. Now, if A is any matrix of this form, it is clear that $\det(\mu I-A)=\det(\mu I-\tilde{A})$, where $\tilde{A}$ is the same as A with $C=0$, i. e., $\tilde{A}$ is symplectic. But if $\tilde{A}$ is symplectic, we have $P_k(A)=0$ for any elementary symmetric function, since k is odd. Hence we obtain also $\lambda(\Gamma^1)(P)_x=0$. q.e.d.

Next we want to check that the Poisson cohomology class of $\lambda(\Gamma^1,\Gamma^0)(P)$ is independent of the connections used to define it.

Given connections $\Gamma^0,\Gamma^1,\Gamma^2$ we consider the family of connections $\Gamma^{s,t}$ whose connection vector fields are $\Lambda^{s,t}_j=(1-s-t)\Lambda^0_j+s\Lambda^1_j+t\Lambda^2_j$, where (s,t) vary in the standard 2-simplex Δ_2. We introduce a $(2k-2)$-vector field $\lambda(\Gamma^2,\Gamma^1,\Gamma^0)(P)$ given by a formula analogous to (4.6) and (4.3):

$$\begin{aligned}&\lambda(\Gamma^2,\Gamma^1,\Gamma^0)(P)\\ &\qquad=k\sum_{\sigma\in S_{2k-2}}(-1)^\sigma\int_{\Delta_2}P(\Lambda^{1,0}_j,\Lambda^{2,0}_j,\Xi^{s,t}_j,\dots,\Xi^{s,t}_j)dtds.\end{aligned}\tag{4.8}$$

Just like in the proof of Proposition 4.1.2, one shows that

$$\delta\lambda(\Gamma^2,\Gamma^1,\Gamma^0)(P) = \lambda(\Gamma^1,\Gamma^0)(P) - \lambda(\Gamma^2,\Gamma^0)(P) + \lambda(\Gamma^1,\Gamma^0)(P). \tag{4.9}$$

Now, we can prove

Proposition 4.2.2. *The Poisson cohomology class* $[\lambda(\Gamma^1,\Gamma^0)(P)]$ *is independent of the connections used to define it.*

Proof. Let Γ^1 and $\tilde{\Gamma}^1$ (resp. Γ^0 and $\tilde{\Gamma}^0$) be basic connections (resp. riemannian connections). It follows from (4.9) that

$$\begin{aligned}\lambda(\Gamma^1,\Gamma^0)(P) - \lambda(\tilde{\Gamma}^1,\tilde{\Gamma}^0)(P) =& \delta\lambda(\tilde{\Gamma}^1,\Gamma^0,\tilde{\Gamma}^0)(P) - \delta\lambda(\Gamma^1,\tilde{\Gamma}^1,\Gamma^0)(P) \\ &+ \lambda(\tilde{\Gamma}^1,\Gamma^1)(P) - \lambda(\Gamma^0,\tilde{\Gamma}^0)(P).\end{aligned}$$

Hence, it is enough to show that the Poisson cohomology classes of $\lambda(\tilde{\Gamma}^1,\Gamma^1)(P)$ and $\lambda(\Gamma^0,\tilde{\Gamma}^0)(P)$ are trivial.

Consider first the basic connections $\tilde{\Gamma}^1$ and Γ^1. The linear combination $\Gamma^{1,t} = (1-t)\Gamma^1 + t\tilde{\Gamma}^1$ is also a basic connection. If $x \in M$, we fix splitting coordinates (x^j, y^k) around x as in the proof of Proposition 4.2.1. Then we see that, with respect to the basis $\{dx^j, dy^k\}$, the matrix representations of $D^1_\alpha - \tilde{D}^1_\alpha$ and $R^t(\alpha,\beta)$ are of the form (4.7). Hence, we conclude that if $P \in I^k(GL(m,\mathbb{R}))$, with k odd,

$$P(\tilde{D}^1_{\alpha_1} - D^1_{\alpha_1}, R^t(\alpha_2,\alpha_3),\ldots,R^t(\alpha_{2k-2},\alpha_{2k-1})) = 0.$$

Therefore, $\lambda(\tilde{\Gamma}^1,\Gamma^1)(P) = 0$, whenever $\tilde{\Gamma}^1$ and Γ^1 are basic connections.

Now consider the riemannian connections Γ^0 and $\tilde{\Gamma}^0$. The linear combination $\Gamma^{0,t} = (1-t)\tilde{\Gamma}^0 + t\Gamma^0$ is also a riemannian connection. All these connections are induced from covariant riemannian connections ∇^0, $\tilde{\nabla}^0$ and $\nabla^{0,t}$, and we can define a $(2k-1)$-form $\lambda(\nabla^0,\tilde{\nabla}^0)(P)$ by a formula analogous to (4.3). Moreover, this form is closed (because k is odd), and $\#\lambda(\nabla^0,\tilde{\nabla}^0)(P) = \lambda(\Gamma^0,\tilde{\Gamma}^0)(P)$. It follows from the homotopy invariance of $H^*(M)$, as in the usual theory of characteristic classes of foliations (see [1], page 29), that

$$[\lambda(\nabla^0,\tilde{\nabla}^0)(P)] = [\lambda(\nabla^0,\nabla^0)(P)] = 0.$$

Hence, the Poisson cohomology class $[\lambda(\nabla^0,\tilde{\nabla}^0)(P)]$ vanishes. q.e.d.

Remark 4.2.3. The assumption that the riemannian connections are of the special form $\nabla_{\#\alpha}$ was used in the proof to invoke the homotopy invariance of $H^*(M)$. Poisson cohomology $H^*_\Pi(M)$ is not homotopy invariant, so in defining the invariant $\lambda(\Gamma^1, \Gamma^0)(P)$ we cannot consider an arbitrary riemannian contravariant connection Γ^0. On the other hand, as we pointed out above, in general a Poisson manifold does not admit a Poisson connection of the form $\nabla_{\#\alpha}$. Hence, the basic connections are "genuine" contravariant connections, i. e., not induced by any covariant connection.

We define the *secondary characteristic classes* $\{m_k(M)\}$ of a Poisson manifold to be the Poisson cohomology classes

$$m_k(M) = [\lambda(\Gamma^1, \Gamma^0)(P_k)] \in H^{2k-1}_\Pi(M), \qquad (k = 1, 3, \dots). \tag{4.10}$$

If M is a symplectic manifold then these classes obviously vanish. They also vanish if $M = S \times N$ where S is symplectic and N has the zero Poisson bracket. However, they do not vanish for a general, regular, Poisson manifold (see the examples below). Hence these characteristic classes give information on both the Poisson geometry and the topology of the symplectic foliation of M. In the next section we give some explicit computations of these classes, and in the following section we will show that the first class coincides with the modular class of M (up to a scalar factor).

Remark 4.2.4. In general, one can only define the characteristic classes m_k for k odd. Assume, however, that M admits flat riemannian connections and flat basic connections (we will see some non-trivial examples below). Then the proofs of Propositions 4.2.1 and 4.2.2 can be carried through, in the class of flat connections, for *any* k. Hence, in this case, one can define characteristic classes m_k for *any* k.

4.3 Examples

We give a few types of Poisson manifolds where one can compute some of the secondary characteristic classes.

Euclidean spaces. Consider a Poisson manifold $M \simeq \mathbb{R}^m$, so we have global coordinates $(x^1, \dots, x^m)$. To compute $\lambda(\Gamma^1, \Gamma^0)(P)$ we take as Γ^0 the flat connection determined by these global coordinates, and as Γ^1 we take the basic connection defined by

$$D_{dx^i} dx^j = [dx^i, dx^j] = \sum_k \frac{\partial \pi^{ij}}{\partial x^k} dx^k.$$

Since $P_1(A) = \frac{1}{2\pi}\mathrm{tr}\,(A)$, we find immediately that the first characteristic class is

$$m_1(M) = \frac{1}{2\pi}\sum_{i,j}\frac{\partial \pi^{ij}}{\partial x^j}\frac{\partial}{\partial x^i}. \tag{4.11}$$

To compute the second characteristic class, we note that $D^t_{dx^i}dx^j = (1-t)D_{dx^i}dx^j$, and we compute its curvature:

$$R^t(dx^i, dx^j)dx^k = -t(t-1)D_{[dx^i,dx^j]}dx^k.$$

Now,

$$P_3(A,B,C) = \frac{1}{24\pi^3}\Big[\mathrm{tr}\,(ABC) - \frac{1}{2}(\mathrm{tr}\,A\ \mathrm{tr}\,(BC) + \mathrm{tr}\,B\ \mathrm{tr}\,(CA) \\ +\mathrm{tr}\,C\ \mathrm{tr}\,(AB)) - \frac{1}{2}\mathrm{tr}\,A\ \mathrm{tr}\,B\ \mathrm{tr}\,C\Big]$$

and the expression for the characteristic class $m_3(M)$ is a certain homogeneous polynomial of degree 5 involving the derivatives of order ≤ 3 of the components π^{ij} of the Poisson tensor.

Linear Poisson structures. Let $M = \mathfrak{g}^*$ with the Lie-Poisson structure determined by the Lie algebra $\mathfrak{g}$. Then, from the previous example, we see that the first class is represented by the constant vector field

$$m_1(\mathfrak{g}^*)(v) = \frac{1}{2\pi}\mathrm{tr}\,(\mathrm{ad}\,v).$$

In this case both the basic connection and the riemannian connection are flat and so we can consider the classes m_k for *any* k. The computations simplify considerably, and we see that all classes can be represented by constant multivector fields. For example, a straightforward computation shows that

$$m_2(\mathfrak{g}^*)(v_1,v_2,v_3) = \frac{3!}{4\pi^2}K_2(v_1,[v_2,v_3]),$$
$$m_3(\mathfrak{g}^*)(v_1,\dots,v_5) = \frac{1}{8\pi^3}\sum_{\sigma\in S_5}K_3(v_{\sigma(1)},[v_{\sigma(2)},v_{\sigma(3)}],[v_{\sigma(4)},v_{\sigma(5)}])$$

where we have set

$$K_j(v_1,\dots,v_j) = \mathrm{tr}\,(\mathrm{ad}\,v_1\cdots\mathrm{ad}\,v_j).$$

Note that K_2 is just the killing form.

In this case it is possible to give a general formula for all characteristic classes:

$$m_k(\mathfrak{g}^*)(v_1,\dots,v_{2k-1}) = \frac{1}{(2\pi)^k}\sum_{\sigma\in S_{2k-1}} K_k(v_{\sigma(1)},[v_{\sigma(2)},v_{\sigma(3)}],\dots,[v_{\sigma(2k-2)},v_{\sigma(2k-1)}]).$$

The proof of these formulas involves a certain amount of computation using Newton's identities for the elementary symmetric polynomials.

Incidentally, we note that the classes m_k are ad-invariant since each K_j is an ad-invariant multilinear form. Therefore, the classes $m_k(\mathfrak{g}^*)$ represent certain cohomology classes in the Lie algebra cohomology of $\mathfrak{g}$.

Poisson-Lie Groups. Let G be a connected Poisson-Lie group (see, e. g., [8]). Then the set of left invariant 1-forms $\Omega^1_{\mathrm{Inv}}(G)$ is closed for the Lie bracket defined by the Poisson bracket. Hence we can define a basic connection D^1 in G by requiring that

$$D_\alpha\beta = [\alpha,\beta], \qquad \forall\alpha,\beta\in\Omega^1_{\mathrm{Inv}}(G). \tag{4.12}$$

This connection is flat.

Let $D^0 = \nabla_{\#\alpha}$ be the unique left invariant connection in G which for left invariant vector fields is given by

$$\nabla_X Y = [X,Y], \qquad \forall X,Y\in\mathfrak{g}.$$

This connection is also flat.

We compute $\lambda(D^1,D^0)(P)$ and, generalizing the previous example, the classes $m_k(G)$ are all represented by the left invariant multivector fields:

$$m_k(G)(\alpha_1,\dots,\alpha_{2k-1}) = \frac{1}{(2\pi)^k}\sum_{\sigma\in S_{2k-1}} K_k(\alpha_{\sigma(1)},[\alpha_{\sigma(2)},\alpha_{\sigma(3)}],\dots,[\alpha_{\sigma(2k-2)},\alpha_{\sigma(2k-1)}])$$

where $\alpha_1,\dots,\alpha_n\in\Omega^1_{\mathrm{Inv}}(G)$. In these formulas, $[\ ,\ ]$, ad and K_k are relative to the Lie algebra $\mathfrak{g}^* = \Omega^1_{\mathrm{Inv}}(G)$.

Remark 4.3.1. Note that if the Poisson bracket in G is not trivial, the contravariant connection defined by (4.12) is *not* left invariant, because left translation in the group is not a Poisson map. These type of connections are studied in a complement to the present paper, where we deal with invariant connections ([3]).

Regular Poisson manifolds. Let M be a regular Poisson manifold of dimension m and corank q. First choose some riemannian connection determining a splitting

$$T^*(M) = T^*(\mathcal{S}) \oplus \nu^*(\mathcal{S}),$$

where $T^*(\mathcal{S})$ (resp. $\nu^*(\mathcal{S})$) is the cotangent (resp. conormal) bundle to the symplectic foliation. We have a riemannian connection D^0 such that:

$$D^0_\alpha(\beta+\gamma) = \nabla^{0,\|}_{\#\alpha}\beta + \nabla^{0,\perp}_{\#\alpha}\gamma,$$

where β and γ, are sections of $T^*(\mathcal{S})$ and $\nu^*(\mathcal{S})$, and $\nabla^{0,\|}$ and $\nabla^{0,\perp}$, are covariant riemannian connections in these bundles.

Because M is regular, we can also choose a covariant connection $\nabla^{1,\|}$ on TM such that $\nabla^{1,\|}\Pi = 0$. We define the basic connection D^1 on M by setting

$$D^1_\alpha(\beta+\gamma) = \nabla^{1,\|}_{\#\alpha}\beta + \nabla^{1,\perp}_{\#\alpha}\gamma,$$

where $\nabla^{1,\perp}$ is a basic connection in $\nu(\mathcal{S})$ in the usual sense of foliation theory (see [1], p. 33). A computation shows that

$$\lambda(D^1, D^0)(P) = \#\lambda(\nabla^{1,\perp}, \nabla^{0,\perp})(\tilde{P}),$$

where $\tilde{P}$ is the obvious restriction of $P \in I^*(GL_m(\mathbb{R}))$ to $I^*(GL_q(\mathbb{R}))$.

It is well known in foliation theory (see [1], p. 66) that the forms

$$\begin{aligned} c_k &= \lambda(\nabla^{1,\perp})(\tilde{P}_k), \qquad (1 \le k \le q) \\ h_{2k-1} &= \lambda(\nabla^{1,\perp}, \nabla^{0,\perp})(\tilde{P}_{2k-1}), \qquad (1 \le 2k-1 \le q), \end{aligned}$$

satisfy

$$dc_k = 0, \qquad (1 \le k \le q) \tag{4.13}$$

$$dh_{2k-1} = c_{2k-1}, \qquad (1 \le 2k-1 \le q). \tag{4.14}$$

and so they can be used to define a homomorphism of graded algebras

$$H^*(WO_q) \to H^*(M),$$

where $H^*(WO_q)$ is the relative Gelfand-Fuks cohomology of formal vector fields in $\mathbb{R}^q$. This homomorphism is independent of the connections and its image are the exotic or secondary characteristic classes of foliation theory.

In this respect, the Poisson secondary characteristic classes are simpler than the corresponding ones in foliation theory: the $(2k-1)$-forms $\lambda(\nabla^{1,\perp}, \nabla^{0,\perp})(\tilde{P}_k)$ are not closed in general, but are closed along the symplectic leaves, so its image under $\#$ is a closed $(2k-1)$-vector field and, hence, define Poisson cohomology classes. Therefore, one has

$$m_{2k-1}(M) = [\# h_{2k-1}] \tag{4.15}$$

but, in general, m_{2k-1} is not in the image of $\# : H^*(M) \to H^*_{\Pi}(M)$.

Still, one can sometimes relate the two types of secondary characteristic classes. Take, for example, the Godbillon-Vey class which by definition is the cohomology class $w = [h_1 c_1^q] \in H^{2q+1}(M)$ (it follows from relations (4.13) that $d(h_1 c_1^q) = c_1^{q+1} = 0$, so $h_1 c_1^q$ does define a cohomology class).

Proposition 4.3.2. *If a regular Poisson manifold has a non-trivial Godbillon-Vey class then it has a non-trivial first Poisson secondary characteristic class.*

Proof. If $m_1(M) = [\# h_1]$ is trivial, we have $\# h_1 = \# df$ for some smooth function f, i. e., $h_1(\#\alpha) = df(\#\alpha)$. But h_1 is defined up to a 1-form in the differential ideal that gives the symplectic foliation, so $h_1 \wedge (dh_1)^q = 0$ and the the Godbillon-Vey class must vanish. q.e.d.

On the other hand, it is perfectly possible for the Godbillon-Vey class to vanish while $m_1(M) \neq 0$. One such example is provided by the Reeb foliation in S^3 with the leafwise area form (see [4, 12] for details on this example).

Another consequence of this relationship is that, for a regular Poisson manifold M, the characteristic classes $m_k(M) = 0$, for $2k-1 > q = \operatorname{corank}(M)$.

As a special case, let us consider a Poisson manifold of corank 1. The only non-vanishing class is $m_1(M)$. If the symplectic foliation is transversely orientable, let Z be a trivializing section of the normal bundle. Let θ be the corresponding 1-form that trivializes the conormal bundle. There exists a 1-form η such that

$$d\theta = \eta \wedge \theta.$$

For $\nabla^{1,\perp}$ we choose a basic connection in $\nu(\mathcal{S})$ such that

$$\nabla^{1,\perp}_X Z = \eta(X)Z.$$

For $\nabla^{0,\perp}$ we choose a riemannian connection such that

$$\nabla^{0,\perp}_X Z = 0.$$

These choices give

$$\lambda(\nabla^{1,\perp}, \nabla^{0,\perp})(\mathrm{tr}\,) = \eta,$$

so we conclude that

$$m_1(M) = \frac{1}{2\pi}[\#\eta].$$

In fact, in this case we have $h_1 = \frac{1}{2\pi}\eta$ so $w = \frac{1}{4\pi^2}\eta \wedge d\eta$ represents the Godbillon-Vey class.

If the symplectic foliation is not transversely orientable one can pass to a double cover and apply the same reasoning.

4.4 The Modular Class

The modular class of a Poisson manifold is an obstruction lying in the first Poisson cohomology group $H^1_\Pi(M)$ to the existence of a transverse invariant measure (see [12] for details on the modular class). It can be defined as follows: Let μ be any measure in M with associated divergence operator $\mathrm{div}_\mu X \equiv \mathcal{L}_X\mu/\mu$. Then one checks that the map $f \mapsto \mathrm{div}_\mu \# df$ is a derivation of $C^\infty(M)$ so defines a vector field v_μ, called the *modular vector field* associated with the measure μ. This vector field is an infinitesimal automorphism of Π. If $\mu' = a\mu$ is another measure, we have $v_{\mu'} = v_\mu + \#d\log a = v_\mu + \delta \log a$, so in fact the modular class

$$\mathrm{mod}\ (M) \equiv [v_\mu] \in H^1_\Pi(M)$$

is well defined and independent of μ.

The examples in the previous section when compared to the computations of the modular class done in [12] suggest the following

Theorem 4.4.1. *For any Poisson manifold M*

$$m_1(M) = \frac{1}{2\pi} mod\ (M). \tag{4.16}$$

Proof. Choose a basic connection D^1 and a riemannian connection D^0 relative to some metric on M. Let μ be the measure defined by this metric. We claim that

$$\lambda(D^1, D^0)(\mathrm{tr}\,) = v_\mu, \tag{4.17}$$

so (4.16) follows.

Observe that it is enough to show that (4.17) holds on the regular points of M, since the set of regular points is an open dense set and both sides are smooth vector fields on M. So assume that $x \in M$ is a regular point and pick Darboux coordinates $(x^1, \dots, x^m)$. If $g = \big(\langle dx^i, dx^j\rangle\big)$ is the $m \times m$-matrix of inner products of the dx^i's, we have

$$\mu = (\det g)^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^m.$$

As in the proofs of the previous section, relative to the basis $\{dx^1, \dots, dx^m\}$, the operator D^1_α has a matrix representation by a traceless matrix, so we only need to understand what is the matrix representation, relative to this basis, of the riemannian connection $D^0_\alpha = \nabla^0_{\#\alpha}$.

Since ∇^0 is a metric connection, parallel transport preserves the volume, and we have for any smooth function $f \in C^\infty(M)$:

$$\begin{aligned}
0 =& \nabla^0_{\#df}\mu \\
=& \#df((\det g)^{\frac{1}{2}})dx^1 \wedge \cdots \wedge dx^m + \\
&+ (\det g)^{\frac{1}{2}}(\nabla^0_{\#df}dx^1 \wedge \cdots \wedge dx^m + \cdots + dx^1 \wedge \cdots \wedge \nabla^0_{\#df}dx^m) \\
=& \left(\#df((\det g)^{\frac{1}{2}}) + (\det g)^{\frac{1}{2}} \mathrm{tr}\, \nabla^0_{\#df}\right) dx^1 \wedge \cdots \wedge dx^m.
\end{aligned}$$

So we conclude that:

$$\mathrm{tr}\,(D^1_{df} - D^0_{df})\mu = \#df((\det g)^{\frac{1}{2}})dx^1 \wedge \cdots \wedge dx^m. \tag{4.18}$$

Now recall that $(x^1, \dots, x^m)$ were Darboux coordinates around a regular point, so the form $dx^1 \wedge \cdots \wedge dx^m$ is preserved by the hamiltonian flows, and we have

$$\mathcal{L}_{\#df}(dx^1 \wedge \cdots \wedge dx^m) = 0.$$

Hence, we conclude that:

$$\mathcal{L}_{\#df}\mu = \#df((\det g)^{\frac{1}{2}})dx^1 \wedge \cdots \wedge dx^m. \tag{4.19}$$

Comparing (4.18) and (4.19) gives

$$\operatorname{tr}(D^1_{df} - D^0_{df}) = \operatorname{div}_\mu \# df,$$

so relation (4.17) holds. q.e.d.

If $(\gamma(t), \alpha(t))$, $t \in [0,1]$, is a cotangent path and X is a vector field, one defines the integral

$$\int_{(\gamma,\alpha)} X = -\int_0^1 i_{X(\gamma(t))}\alpha(t)dt.$$

(For basic properties of this integral see [4]). As a corollary of the theorem and the Ginzburg and Golubev formula (3.6), we obtain:

Corollary 4.4.2. *Let (γ, α) be a cotangent loop in the symplectic leaf S. Then*

$$\det H^L_S(\gamma,\alpha) = \exp\left(\int_{(\gamma,\alpha)} tr(D^1 - D^0)\right), \tag{4.20}$$

where the determinant is relative to the transverse measure induced by the volume element of the metric associated with D^0.

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